International Journal of Mathematical Education in Science and Technology, Vol. 36, Nos. 2–3, 2005, 219–235

Taylor & Francis
Taylor & Francis Group

Mathematical discovery and *affect*: the *effect* of AHA! experiences on undergraduate mathematics students

PETER G. LILJEDAHL

Faculty of Education, Simon Fraser University,
8888 University Dr., Burnaby B.C., V5A 1S6, Canada

(*Received August 2004*)

The AHA! experience – the moment of illumination on the heels of lengthy, and seemingly fruitless, intentional effort – has long been the basis for lore in mathematics. Unfortunately, such lore is often restricted to the discussion of these phenomena in the context of great mathematicians and great mathematical advancement. But are such experiences reserved only for the upper echelons of mathematical practice? This study focuses on the impact of these AHA! experiences on undergraduate mathematics students' affective domain. In particular, the role of the positive emotion that accompanies such moments of illumination in changing the attitudes and beliefs of 'resistant' students is examined. That is, pre-service elementary school teachers who deem themselves to be incapable and/or phobic of mathematics and the learning of mathematics but are forced to take an undergraduate mathematics course as qualification for entry into a teacher education programme. The results indicate that an AHA! experience has a transformative effect on 'resistant' students' affective domains, creating positive beliefs and attitudes about mathematics as well as their abilities to do mathematics.

1. Introduction

> Perhaps I could best describe my experience of doing mathematics in terms of entering a dark mansion. One goes into the first room, and it's dark, completely dark. One stumbles around bumping into the furniture, and gradually, you learn where each piece of furniture is, and finally, after six months or so, you find the light switch. You turn it on, and suddenly, it's all illuminated.
>
> Andrew Wiles[1]

Suddenly, it's all illuminated. In the time it takes to turn on a light the answer appears and all that came before it makes sense. A problem has just been solved, or a new piece of mathematics has been found, and it has happened in a flash of insight, in a moment of illumination, in an AHA! experience. From Archimedes to

Email: liljedahl@sfu.ca
[1]From the movie *The Proof* (Nova [1]).

International Journal of Mathematical Education in Science and Technology
ISSN 0020–739X print/ISSN 1464–5211 online
http://www.tandf.co.uk/journals
DOI: 10.1080/00207390412331316997

Andrew Wiles, from research mathematician to amateur mathematician, the AHA! experience is an elusive, yet real, part of 'doing' mathematics. Although it defies logic and resists explanation, it requires neither logic nor explanation to define it. The AHA! experience is self-defining. At the moment of insight, in the flash of understanding when everything seems to make sense and the answer is laid bare before you, you know it, and you call out – AHA!, I GOT IT! However, the AHA! experience is more than just this moment of insight. It is this moment of insight on the heels of lengthy, and seemingly fruitless, intentional effort [2]. It is the turning on the light after six months of groping in the dark.

Simply put, the AHA! experience is the experience of having an idea come to mind with 'characteristics of brevity, suddenness, and immediate certainty' ([3], p. 54). It is the phenomenon of 'sudden clarification' ([4], p. 54) arriving in a 'flash of insight' ([5], p. 283) and accompanied by feelings of certainty [6–8].

Literature is rich with examples of these AHA! experiences – from Amadeus Mozart's seemingly effortless compositions [2] to Samuel Taylor Coleridge's dream of Kubla Kahn [9], from Leonardo da Vinci's ideas on flight [10] to Albert Einstein's vision of riding a beam of light [9] – all of which exemplify the role of this elusive mental process in the advancement of human endeavours. In particular, scientific advancements are often associated with these flashes of insight, bringing forth new understandings and new theories in the blink of an eye. Furthermore, the moment of illumination – the AHA! experience – has long been the basis for lore in mathematics. Unfortunately, such lore is often restricted to the discussion of these phenomena in the context of great mathematicians and great mathematical advancement. But are such experiences reserved only for the upper echelons of mathematical practice? This study focuses on the impact of these AHA! experiences on undergraduate mathematics students' affective domain. In particular, I examine the role of the positive emotion that accompanies such moments of illumination in changing the attitudes and beliefs of 'resistant' students. That is, pre-service elementary school teachers who deem themselves to be incapable and/or phobic of mathematics and the learning of mathematics but are forced to take an undergraduate mathematics course as qualification for entry into a teacher education programme.

2. Prologue (motivation and question)

Imagine a person who has a dislike for mathematics. Somewhere in their past is a negative experience with the subject – it may have been a single event, or it may have been a series of negative events. The nature of the experience is not clear, but what is clear is the impact that it has had on them as a learner of mathematics. They feel that they 'were never good at mathematics', or that they 'can't do mathematics'. Perhaps they are even afraid of the subject, suffering anxiety at the thought of having to endure a mathematics course.

Such a person is not difficult to imagine. We encounter such individuals on a regular basis. They are our neighbour, our brother, our friend – and sometimes they are our student. Now imagine a whole class of such students – 100 plus individuals who, to a person, would describe themselves as either being incapable of doing mathematics, or having a phobia about the learning of mathematics, or both.

What impact would an AHA! experience have on the individuals in such a group? These are not mathematicians – yet they are working in the field of mathematics.

They are not anticipating these moments of insight; they may not even be familiar with such moments. For many of them mathematics has long been a subject devoid of wonder, surprise, and discovery. As a researcher I became interested in what such a population would make of these moments of illumination in the context of mathematics. Were they experiencing them, and if so, what effect were they having on them?

3. Background

> In that moment when the connection is made, in that synaptic spasm of completion when the thought drives through the red fuse, is our keenest pleasure.
>
> Thomas Harris [11, p. 132]

The learning of mathematics has classically been studied from the perspective of cognition. That is, the examination of what are the cognitive processes involved in learning, and how do they operate. However, the inability of such research to explain the failures of people in problem solving contexts who possess the cognitive resources necessary to succeed has prompted the re-evaluation of the role of the affective domain in the learning of mathematics [12].

The affective domain is most simply described as *feelings* – the feelings that students have about mathematics. In general it is understood that the affective domain is composed of three dimensions: beliefs, attitudes, and emotions [13]. The beliefs are just that—what students believe; what they believe to be true about mathematics and what they believe about their ability to do mathematics. Beliefs about mathematics are often based on their own experiences with mathematics. For example, beliefs that mathematics is 'difficult', 'useless', 'all about one answer', or 'all about memorizing formulas' stem from experiences that have first introduced these ideas and then reinforced them. Research has shown that such beliefs are slow to form in a learner, and once established, are equally slow to change, even in the face of intervention [14].

A qualitatively different form of belief is with regards to a person's beliefs in their ability to do mathematics, often referred to as efficacy, or self-efficacy. Self-efficacy, like the aforementioned belief structures, is a product of an individual's experiences with mathematics, and is likewise slow to form and difficult to change. Self-efficacy with regards to mathematics has most often been dealt with in the context of negative belief structures [15] such as 'I can't do math', 'I don't have a mathematical mind', or even 'girls aren't good at math'.

Attitudes can be defined as 'a disposition to respond favourably or unfavourably to an object, person, institution, or event' [16, p. 4]. Attitudes can be thought of as the responses that students have to their belief structures. That is, attitudes are the manifestations of beliefs. For example, beliefs such as 'math is difficult', 'math is useless', or 'I can't do math' may result in an attitude such as 'math sucks'. A belief that 'math is all about formulas' may manifest itself as an attitude of disregard for explanations in anticipation of the eventual presentation of a formula. Attitudes, like beliefs, are stable entities; they are slow to form and difficult to change.

Emotions, on the other hand, are relatively unstable [14] and, as a result, the role that they play in the learning of mathematics has received little attention (for exception see [17, 18]). They are rooted more in the immediacy of a situation or a task and as a result are often fleeting. Students with generally negative beliefs and attitudes can experience moments of positive emotions about a task at hand or, conversely, students with generally positive outlooks can experience negative emotions. Changes in beliefs and attitudes are generally achieved through the emotional dimension – repeated negative experiences will eventually produce negative beliefs and attitudes, and likewise, repeated positive experiences will produce positive beliefs and attitudes. However, as mentioned earlier, existing literature indicates that change is slow.

Because of their stable nature, research has focused primarily on the role of beliefs and attitudes on the learning of mathematics [14]. The results indicate that beliefs and the resulting attitudes are strongly linked to school achievement [19, 15]. They are the gatekeepers to learning. Before a student can even begin to engage in mathematical content they have to first decide that they are both capable of learning the presented material, and willing to do so. Once this has been decided any residual beliefs and attitudes regarding the learning of the content will continue to affect their learning of it. For example, a student with the belief that mathematics is about 'memorizing rules' will approach new content matter from the perspective of identifying a rule, mastering the use of that rule, and then memorizing that rule – regardless of the intended cognitive outcome of the lesson.

4. Methodology

The participants for this study are undergraduate students at Simon Fraser University enrolled in a *Foundations of Mathematics for Teachers* course (MATH 190). At the time of the study this course was one of a number of courses that could be taken to satisfy a mathematics prerequisite for entry into a teacher education programme for prospective elementary school teachers. However, of all the options available, enrolment in MATH 190 was by far the most common route towards satisfying this requirement and as such was populated almost entirely by prospective teachers.

This course had been designed with the intention of providing its enrollees with a foundational understanding of elementary school mathematics. There is a focus on conceptual understanding of topics as opposed to an ability to replicate procedural algorithms. There is an attempt to look at specific strands of mathematics such as geometry and number theory in their entirety as opposed to the piecewise and fragmented way in which mathematics is often experienced in a spiralled curriculum. There is also an attempt to integrate an underlying appreciation for mathematical thinking and reasoning across all strands of the course. To put it simply, MATH 190 is regarded as an 'unpacking' (or 'repacking') course. It lays out all of elementary mathematics at one time for examination and conceptual reorganization. The hope is that such an approach would allow the students to build connections between individual strands of mathematics and facilitate a deeper understanding of the topics as they make sense of all that they see. The course runs for the length of one semester (13 weeks) and has four contact hours each week. The course mark is determined from performance on weekly assignments (10 in total), a project, two midterm exams, and a final.

The course is also specifically delivered with the enrollees, themselves, in mind. In general, students enrolled in MATH 190 are best described as *resistant*. That is, they are resistant to the fact that they have to take a mathematics course. Many of the students would describe themselves as either being math-phobic, math-incapable, or a combination of the two. They usually have negative beliefs about their abilities to do mathematics, poor attitudes about the subject, and dread the thought of having to take a mathematics course. Such being the case, there is a great effort made to alleviate some of these anxieties in the pedagogical approaches to the course. This is more than an emphasis on making the course content seem less daunting, although such an emphasis is certainly made. For example, students are expected to complete their assignments in groups of three to five members. It is felt that in addition to group work being a positive method for learning it is also a structure for support and encouragement. Students are also provided with support through an open tutorial lab. The lab is open to students from 20 to 30 hours per week and is staffed at all times with between one and four teaching assistants. This is a place where students can go to seek help should they need it as well as a place to meet with their groups and work on homework assignments as well as an end-of-term project.

The data for this study comes from the end-of-term project, one of the options for which was to write about a mathematical AHA! that they had experienced in their participation in the course. As it was not certain that everyone could claim to have had such an experience the students were offered an alternative to this assignment. They could, if they wished, engage in a mathematical investigation centred around a problem to solve. In order to be fair this option was open to all students regardless of their experience with an AHA! The project was worth 10% of their final mark and they were given four weeks to work on it. The portion of the project that relates to the AHA! experience follows:

> 'I had been working on the problem for a long time without any progress. Then suddenly I knew the solution, I understood, everything made sense. It seemed like it just CLICKED!'
>
> The above anecdote is a testament of what is referred to as an AHA! experience. Have you ever experienced one? The purpose of this assignment is to have you reflect upon such an AHA! experience and to explore exactly what you learned in that instance and what you think contributed to the moment. You will hand in:
>
> 1. A detailed explanation of the specific mathematical topic that you were studying and the difficulty you were having with it (including any incorrect or incomplete understandings that you had of the topic before the AHA!).
> 2. The story of the AHA! experience as you remember it, paying particular close attention to what you were doing before it happened, when it happened, and how it made you feel when it happened.
> 3. A detailed explanation of your new understanding of the mathematical topic.
> 4. A conclusion as to how, upon reflection, the AHA! experience contributes to mathematical learning in general, and for you in particular.
> 5. Anything else that you feel would contribute to the reader gaining insight into the moment as you experienced it.
>
> Your final product will be evaluated for completeness and clarity.

Rather than provide a definition of the AHA! experience, such as the one presented in the first section of this article, participants were instead presented with an anecdotal account using language that was descriptive rather than definitive. This was in part due to the fact that I did not wish to corrupt their responses regarding the nature of their AHA! experience with ideas that would be included in such a definition. It was also due to the fact that, as mentioned earlier, the AHA! experience is self-defining. That is, I did not have to define it for the participants, only cue it. In order to give the participants a broader perspective of the survey they were also provided with a brief statement of the overall purpose of the assignment, in the hope that this would prevent them from getting overly caught up in the details of the individual questions. To access the mathematical context of the AHA! experience, question one focused specifically on the mathematics within which the AHA! occurred. Question two was designed to elicit a response regarding the illumination phase of the experience and focused on the story of the AHA! as well as on the feelings invoked by it. Question three was the continuation of question one and was included in order to measure the change in understanding of a mathematical topic across the AHA! experience. The last two questions were included as a way to allow the participants to present their own ideas on the phenomenon should they have any.

5. Results

Of the 112 students enrolled in the course, 76 students chose to write about their AHA! experience. Of these, 65 recounted such an experience in the context of their course experience. The remaining 11 participants responded to the survey in the context of an AHA! that they had had in their pre-tertiary schooling. I did not dismiss these accounts, but I did analyse them separately.

The 76 responses can be grouped into two categories: teaching and discovering. The first of these, teaching, pertains to AHA!'s experienced in the passive reception of mathematical content. In total, there were 14 accounts of such AHA! experiences. Each of these told of an event in the lecture hall where something the instructor said or demonstrated caused them to understand a previously not understood piece of mathematical content. A much more common AHA! experience, however, was of the discovery type, which accounted for the remaining 62 responses. These were descriptions of AHA!'s that had occurred in the context of trying to work something out for themselves, either in solving a problem or working towards understanding some particular mathematics content. The 11 accounts of AHA! experience that fell outside of the context of the course were also distributed between these two types of AHA!'s with six in the teaching type and five in the discovery type. These results are summarized in table 1.

This table clearly shows the predominance of discovery type AHA!'s over teaching type AHA!'s. This is not surprising given that to discover something presupposes an AHA! experience, whereas to learn something through the passive reception of being taught to does not.

5.1. *An overview of the responses*

Although the main focus of this article is on the affective dimension of the students' AHA! experiences as stemming from their responses to question two (above) I would

Table 1. Undergraduate mathematics students' response rates by AHA! context and AHA! type.

	Teaching AHA!'s		Discovery AHA!'s	
	number	% of n ($n = 76$)	number	% of n ($n = 76$)
AHA!'s within the course	8	11%	57	75%
AHA!'s prior to the course	6	8%	5	7%
Totals	14	18%	62	82%

be remiss if I did not also mention something about their responses to the rest of the questions. From the five questions it is obvious that I was hoping that detailed descriptions of the mathematical context of the AHA! experience would provide me with some insights into the stimuli that invoked the experience. I was also hopeful that descriptions of the newfound understandings of the participants after the AHA! experience, again situated in the particulars of the mathematical context, would provide insight into the cognitive transformation created by the experience. Unfortunately, the data presented account after account of mathematical AHA! experiences in which the cognitive transformation was unremarkable. I present here Kim's comments as an example.

> My AHA! experience came during the first midterm. The question that I was having difficulty with was the very first one. That was already a bad sign. The question was concerning patterns and involved stars arranged in a triangular fashion.

P(1) **
*

P(2) * * *
**
*

P(3) * * **
* * *
**
*

. . . . . .

> The question asked how many stars would be in P(12) and how many stars in P(121)? I was lost. I added up all the stars for each number and compared them to each other and tried to figure out the pattern there. I drew diagrams that made no sense. I tried to use formulas that I had memorized, placing n's and $(n-1)$'s and $(n+1)$'s anywhere, but that was not yielding anything correct. Everything I thought I remembered was completely disintegrating out of my brain and as the minutes ticked I began to panic. As I turned the page there was a similar question as the first. I was lost again. The exam is going very poorly. I ultimately decided to skim throughout the rest of the test and purely answer whatever I could and if I had moments to spare I would go back

> to the puzzling questions. I went back to question one and looked at what I had written down thus far. Then I erased it all. I started to draw more figures representative of the ones given **and then it hits me!** I realized that the number of stars in the first row of any P(n) was one larger than n and that they descended by 1 to 1. That is P(4) starts with 5, then 4, 3, 2, 1. I now remember the very first math class where we discussed patterns and Gauss and how to add all the numbers together up to 100. So, P(12) = 13 + 12 + 11 + ⋯ + 1.
>
> $$\begin{array}{l} 13+12+11+\cdots+1 \\ \underline{1+2+3+\cdots+13} \\ 14+14+14+\cdots+14 = 14\times 13 = 182 \end{array}$$
>
> So, P(12) = 182/2 = 91 (because we counted each twice)
>
> Thus, the answer is 91 stars for P(12). I could now answer the rest of the questions with ease. I began to relax …

As can be seen in Kim's comments, the mathematical understanding that reveals itself through the AHA! experience is not a great leap of understanding but more of a natural (standard) progression as expected from a student working on such a problem. This is very different from the examples of AHA! experiences presented earlier regarding Mozart, Einstein, etc. in which the AHA! is marked not only by a sudden appearance of an idea, but by the sudden appearance of a BIG idea.

In general, all of the students' accounts showed that prior to the AHA! experience there existed in the mind of the participant a piece of mathematics that they either did not understand or a problem that they could not solve. For all intents and purposes they were 'stuck'. After the AHA! experience they now understood this mathematics or could solve the problem. They were 'unstuck'. The AHA! was clearly in the middle of all this and intimately involved in the transition from being 'stuck' to being 'unstuck'. However, at the level of mathematical understanding this transition from 'stuck' to 'unstuck' was minute. It was unremarkable and in many cases indistinguishable from simply having learned something. It was clear that there was an experience of some importance, but that importance was not played out at the level of mathematical understanding. That is, the power of the experience lay in the experience of an answer or an idea arriving in an untimely and unanticipated manner and not in the answer or idea itself [20].

5.2. *Participants' responses*

As already mentioned 76 students chose to write on their AHA! experience. Of these, all but one mentioned how they felt when they experienced the AHA!, even if only to say 'I felt great'. However, not all of the students portrayed an accurate understanding of what was meant by an AHA! experience. Although there were no false AHA!'s – AHA! experiences that proved to be unfounded – there were three cases of misunderstood AHA!'s. One student took the assignment to mean that she was to construct an AHA! experience. This would have been fine if there had, indeed, been a moment of illumination within the process, but her writing indicated that there hadn't. Instead, there was an observable slow awakening to the concept. Evidence of thinking of the AHA! experience as a slow dawning of understanding was also present in the writing of two other students. Andrea even went so far as to rename it the *AAAHA! experience*.

> Andrea: My AHA! came slowly – not all at once, but little by little I grasped the concept.

The remaining 73 students all presented experiences that were consistent with what was expected in the spirit of the assignment (i.e. true AHA! experiences). These responses were recursively coded according to affective themes that were emerging from the data. In what follows I present these themes through the discussion of representative excerpts from students' responses. That is, for each theme I use excerpts that exemplify the themes while at the same time being representative of all of the students' responses pertaining to that theme. These excerpts stand alone, away from the mathematical context in which they occurred. I do this for two reasons. The first is that my initial analysis of the data showed that the mathematical context and the progression of understanding within this context was both unremarkable and unhelpful in the consideration of the affective domain. The second reason is that by presenting these excerpts in their decontextualized form the themes are much more apparent.

5.2.1. Anxiety. Although the topic of anxiety was not brought up in the context of the AHA! experience, 34 students felt it necessary to mention how they felt about mathematics, or about taking a mathematics course. They seemed to do this as a way to provide a baseline for a discussion of their changing feelings. In these 'baseline' discussions the theme of anxiety was prevalent, manifesting itself in terms of dislike, fear, apprehension, and traumatic memories.

Jennifer, Stephanie, and Tonia reflect on how they feel about having to take this course in order to be able to enter into the teaching programme. While Jennifer states a dislike for mathematics, Stephanie and Tonia express a fear of the subject matter.

> Jennifer: I have never been a person that likes or even enjoys math at all, so the idea of having to take this class if I wanted to teach wasn't very appealing to me. So I came into the course with the preconception that it would be just like any other math class that I had taken.
>
> Stephanie: When I entered MATH 190, I felt that fear in my stomach return. I needed this course to enter teaching so the pressure was on.
>
> Tonia: I was scared of the subject as a student and this was magnified 100 times as a teacher. I knew I had to take this course because I did not want my students to feel the same way as I did about math.

Marcie reflects on her experience in mathematics in general – going all the way back to her negative elementary school experiences.

> Marcie: I was feeling emotions that should not have even existed in grade school.

As mentioned earlier, most students are resistant to taking this course. This resistance is by and large due to the anxiety they have towards mathematics. From the informal discussions I had with the students at the beginning of the course I learned that this anxiety, and resulting resistance, was so strong that many of them had deferred taking this course until their last semester of undergraduate studies. Given that this course was a prerequisite for entry into the teaching programme, this 'waiting till the last minute' strategy created a new type of anxiety pertaining to the pressure of having to succeed, and succeed now. This is reflected in Maggie's comment.

> Maggie: I never liked math when I was in school and so I had avoided it when I got to university. I knew I had to have this course in order to apply for PDP [the teacher education programme], but I put it off and put it off until now. It is the only course left for me to take before I can apply for PDP.

5.2.2. Pleasure. All but one of the participants in this study mentioned something about how the AHA! experience made them feel. Although their comments varied in length and details with regard to these feelings, each of them stated in one way or another that it felt 'great'. In what follows I provide a partial list of some of these comments.

> John: It felt great.
>
> Ruth: I was so relieved; I could barely contain my happiness.
>
> Jenny: This was the best feeling.
>
> Christina: I never knew I could feel so good while doing math.
>
> Keri: Wow!
>
> Stacy: The joy I felt was like none other.
>
> Natalie: It made me feel like I could do anything.

It is clear that the AHA! experience produced a positive affective response in the students. However, as will be shown in the next two sections, the AHA! produced more than simply a 'good' feeling. It contributed to a positive change in the beliefs and attitudes of many of the students.

5.2.3. Change in beliefs. Of the 76 students who chose to do their project on their AHA! experience 61 of them discussed their beliefs. Moreover, each of these 61 students did so in the context of changing beliefs. That is, they expressed a change in their beliefs through the experience of the AHA!

Susan describes how the experience has changed her beliefs on both her ability to solve problems and the process she uses to produce a solution.

> Susan: The AHA! experience is inspiring. It makes students believe that they solved that question through reasoning and deep thought, and inspires him or her to seek more of these moments to obtain a sort of confidence and further knowledge.

This was a common theme, often manifesting itself in discussions of newfound confidence as expressed by Steve and Andrea.

> Steve: Initially this course made me very unsure of myself but now I am confident when working out problems among my homework group. Previously, I naturally deferred to them, but after this AHA! experience I got confidence in my answers.

> Andrea: In reflecting upon this AHA! experience I feel a sense of pride that I accomplished this mathematical idea by myself. I am relieved to know that I do not have to depend on others to help me along. This moment also gave me a self-confidence boost in the sense that I may have something to contribute to others, for example my group members.

James reflects on how the absence of these experiences may have contributed to his belief that he was not good at mathematics.

> James: For myself, I wish that I'd had more of these moments in my earlier years of high school then I would maybe not have so readily decided that I was not good at math.

The belief of what 'it takes' to be good at math is altered for Lena as she expresses that she now sees that it is not an issue of intelligence.

> Lena: Knowing that I could stare at a problem and in time I would understand, gave me more confidence that I could be successful in math. It really is not an intelligence issue.

Karen sits on the border between beliefs in her ability to do mathematics and her belief in what it takes to do mathematics.

> Karen: I used to think that if you couldn't get it right away you didn't know how to do it. This is the longest I've ever worked on a problem. I had just about given up when it just came to me. I now know that sometimes it just takes time.

Although Karen's response was similar to that of one other student, her response is unique in that she arrived at this conclusion in the context of doing the other option for the final project. Karen had not intended to write on an AHA! experience and so chose to pursue the problem solving option of the assignment. It was during her work on this problem that she had what she claims to be her first mathematical AHA! experience.

What is interesting is the variety of beliefs that were affected by experiencing illumination in the context of mathematics. Although most of them centre on their own conceptions of their abilities to do mathematics some students expressed how their beliefs about mathematics have changed, as seen in Paula's statement.

> Paula: I used to think that math was all about the right answer, but now I am more aware of the value of the process.

5.2.4. Change in attitudes. Because attitudes are the manifestations of beliefs it was sometimes difficult to distinguish between the two. That is, almost every expression of a change in attitude had a discernable change in beliefs associated with it – and has been counted in the 61 responses discussed above. Charlotte and Stephen express a change in optimism and expectations, respectively.

Charlotte: I have a better attitude now; I'm more optimistic. This is helpful in learning as complete thought processes can be impeded by a dejected attitude.

Stephen: Also, I enjoy math now. I feel like this success stimulated more success. Now I have raised my expectations in math.

Carla has come to terms with her lack of knowledge of mathematics and found within it a new attitude for success.

Carla: I've decided that I really don't know a lot of math. But who cares? I know enough. And I know how to think enough to find the answers. And I know how to ask for help. And I don't care so much about the end result.

However, a few students clearly demonstrate a change in attitude without expressing an obvious change in beliefs. This is best demonstrated in Kristie's comment.

Kristie: I must admit that math is challenging for me ... after the AHA! experience you feel like learning more, because the joy of obtaining the answer is so exhilarating. It almost refreshes one's mind and makes them want to persist and discover more answers. It gave me the inspiration and the determination to do the best that I can do in the subject.

Kristie has most definitely changed her attitude about the pursuit of mathematics in that she is feeling inspired and determined to succeed in the course. What is not clear is whether or not this is as a result of a new belief that she can succeed.

5.3. *Analysis*

Almost all of the participants alluded to a sense of accomplishment that accompanies the AHA! experience, most actually using the word 'accomplishment' to describe the feeling. However, it should be noted that this sense of accomplishment is a secondary result of the AHA! experience, the primary result being the successful solution of a problem or the coming to understand a piece of mathematics. That is to say, a sense of accomplishment comes from accomplishing something. Deanna demonstrates this nicely.

Deanna: After I understood the question and I had completed it, I felt as though I had accomplished something. I felt as though I was somewhat complete in my understanding of the problem.

I make this distinction for one very important reason, to contrast the effect that accomplishment has on beliefs and attitudes with the effect that the AHA! experience has.

There is a wealth of research that indicates that success and feelings of accomplishment contribute to a change in attitudes and beliefs [19, 15]. However, the change they produce is minute. Long periods of sustained and successive success are required to create significant change. This is why beliefs and attitudes are considered to be stable in nature and why positive experiences are claimed to be so important in teaching and learning of mathematics. However, I question this view. The data in this study clearly shows that beliefs and attitudes can be drastically changed through a single AHA! experience. This is not the mark of stability. It may be true that these dimensions of the affective domain resist change in the face of successful completion of mathematical activity, but they yield easily to the phenomena of the AHA! experience. The question remains, however, by what mechanism is such drastic change in the affective domain possible?

I have two possible explanations for this phenomenon. The first is that the positive emotion that is achieved during an AHA! experience is much more powerful than the emotions that are achieved through non-illuminated problem solving. As a result, the effect that they have on beliefs and attitudes is that much more drastic. Furthermore, an AHA! experience often presupposes an accomplishment. Perhaps the sense of accomplishment is heightened and intensified through the mechanism of discovery, once again producing that much bigger change in the affective elements of beliefs and attitudes.

The second explanation has to do with inspiration. Having solved something challenging, or understood something difficult, besides being a great accomplishment is also a measure of what is possible. Success breeds success, and the students seemed to know this. They were inspired to continue, to get better.

> Elizabeth: AHA moments are those great moments of deeper understanding and clarification of problems where incorrect or incomplete understanding is overcome. These moments inspire us and encourage us to keep going despite the frustration and anxiety that often tends to overwhelm us in times of difficulty when attempting to solve a problem.

Elizabeth articulates very nicely that AHA! experiences 'inspire us and encourage us to keep going'. With such motivation success seems to be inevitable. Perhaps it is the anticipation of greater mathematical understanding and ability that changes beliefs and attitudes for the future. This is exemplified in David's optimistic outlook on mathematics and the AHA! experience.

> David: The moment of comprehension is what keeps 'wannabe' mathematicians in the game. The hope that one day, in one instant, the world will mysteriously come into alignment and math will make sense.

6. Conclusions

The moment of illumination, the AHA! experiences, that instance when the connection is made is part of the culture of mathematics. They are the 'fishing tales' that mathematicians tell. But they are not the exclusive property of practising mathematicians. Their power to transform attitudes and beliefs towards the learning of mathematics makes these instances of insight an indispensable resource in the fostering of mathematics students. That they should be taken advantage of is indisputable. The order of business should now be – how to use them? That is, how are we going to orchestrate our students' learning environments to best facilitate the potential for illumination?

7. Afterward

With respect to this final question – how are we going to orchestrate our students' learning environments to best facilitate the potential for illumination? – some work has already been done by the author in this regard. Although far from conclusive, in what follows some of this work is presented.

To begin with, the idea that an AHA! experience can be orchestrated has to be qualified with the fact that such experiences are largely dependent on chance, intrinsic chance and extrinsic chance. Intrinsic chance deals with the luck of coming up with an answer, of having the right combination of ideas join within your mind to produce a new understanding. This was discussed by Hadamard [2] as well as by a host of others under the name of 'the chance hypothesis'. Extrinsic chance, on the other hand, deals with the luck associated with a chance reading of an article, a chance encounter, or some other chance encounter with a piece of mathematical knowledge, any of which contribute to the eventual resolution of the problem that one is working on. However, the idea that mathematical discovery often relies on the fleeting and unpredictable occurrence of chance encounters is starkly contradictory to the image projected by mathematics as a field reliant on logic and deductive reasoning. Ironically, this contributory role of chance emerged from a study done with prominent research mathematicians [20, 21] in which a portion of Hadamard's [2] famous survey was resurrected in order to solicit responses pertaining to illumination, creativity, and insight. The upshot of this strong dependence on chance means that the orchestration of an AHA! experience can best be described as the *occasioning* of an AHA! experience. That is, the environment for such an experience can be orchestrated, but the experience itself cannot.

Also emerging from this study with prominent mathematicians were themes regarding the role of talking, the de-emphasis of details, the importance of re-creation, the need for time, and the role of perseverance. How some of these elements can play out in the context of a classroom is discussed in detail in what follows.

Based on the themes that emerged from the work with the prominent mathematicians I redesigned both the content and the delivery of two *Designs for Learning Mathematics* courses that I was teaching to a group of pre-service teachers. In particular, the way in which problem solving was incorporated into the course was restructured. To begin with there was greater emphasis placed on talking and less emphasis placed on details. This manifested itself primarily in the way the problems were introduced. Instead of providing the students with precisely worded instructions and explanations of the problems, as had been done in the past, the problems were introduced orally, perhaps with some demonstrations and hand gestures.[2] This focus on the oral delivery of problems was problematic, especially for students who were absent, but it was made clear that it was expected and that they should contact a peer for instructions should they miss a session. Students were allowed to take notes if they wished, but were encouraged to do so only after they had played around with the problem.

This focus on talking also played a role in the use of group work. Group work is something that I have always valued and had always used. However, given the mathematicians' comments regarding talking I decided to place even greater emphasis on group discussion and peer interaction. One way in which this was accomplished was through the increased provision of class time to work together on problems. An altogether different way of facilitating interaction, however, was to physically remove myself from any peer discussions centred on the problems that had been given out. I knew from past experience that my presence changed the nature of conversations from hypothesising to questioning. That is, the students would stop talking to each other and start asking me to validate their claims. As such, an explicit effort was made to remove myself from peer interactions as soon as such requests for validation arose, sometimes even leaving the room. Of course, this made observation of any AHA! experiences extremely difficult, but I was confident that my extensive use of journalling[3] would capture any such experiences for later analysis.

Another adjustment that was made had to do with time. This was primarily in the form of much more class time to work on problems and came in two forms: time immediately after being assigned the problem, and time to revisit already assigned problems. As already mentioned, both of these allocations were provided in order to create opportunities for discussion. However, they also served the purpose of creating an interval like re-visitation of problems mentioned by the mathematicians. One further use of time was incorporated; deadlines were extended as much as possible. That is, I allowed much more time to work on problems than I had ever done in the past. This is in keeping with the idea that problem solving can take an extremely large amount of time, and to be done well, an extremely large amount of time should be provided. The final change made in the delivery of the course has, once again, to do with the role of extrinsic chance. By constantly filling the air with relevant, but not explicitly linked, information I increased the opportunity for a chance encounter to occur.

The end results were encouraging. In both classes in which this was attempted significant number of AHA!'s were experienced. Of the 38 pre-service elementary

---

[2]Of course, there were a few exceptions to this. If the details of the problem were so intricate that they could not be remembered the problem was provided in writing. Also, if there were details that were beyond the scope of getting into the problem, such as submission guidelines and instructions pertaining to journalling, then written instructions of those details were also provided.

[3]The students journaled about their problem solving work, in general, and their AHA! experiences in particular using two very different types of journalling techniques. The first type is a simple reflective journal in which they respond to prompts such as: tell me about your problem solving techniques, reflect upon any AHA! experiences that you may have had, what makes a good problem, etc. The second type of journalling is more real-time and is based on the writing style of Douglas Hofstadter [22] in which the writer adopts a trinity of personas to more accurately convey their efforts, failures, and successes in solving challenging problems [23].

teachers taking the course 36 (95%) claimed to have had at least one AHA! experience in their journalling. Of the 34 pre-service secondary teachers taking course 29 (85%) also had clear and discernable evidence of the experiences in their journalling. Although there are too many uncontrolled variables to make any definitive claims about the occasioning of an AHA! my personal experience in teaching these courses tells me that there were far more such experiences than in previous incarnations of the course.

References

[1] Nova, 1993, *The proof*. Aired on PBS on October 28, 1997. Retrieved December 11, 2003 from: http://www.pbs.org/wgbh/nova/transcripts/2414proof.html.
[2] Hadamard, J., 1954, *The Psychology of Invention in the Mathematical Field* (New York: Dover Publications).
[3] Poincaré, H., 1952, *Science and Method* (New York: Dover Publications).
[4] Pólya, G., 1965/1981, *Mathematical Discovery: On Understanding, Learning and Teaching Problem Solving*, Vol. 2 (New York: Wiley).
[5] Davis, P. and Hersh, R., 1980, *The Mathematical Experience* (Boston, MA: Birkhauser).
[6] Burton, L., 1999, The practices of mathematicians: What do they tell us about coming to know mathematics? *Educational Studies in Mathematics*, **37**(2), 121–143.
[7] Fischbein, E., 1987, *Intuition in Science and Mathematics: an Educational Approach* (Dordrecht: Kluwer).
[8] Rota, G., 1997, *Indiscrete Thoughts* (Boston, MA: Birkhauser).
[9] Ghiselin, B., 1952, *The Creative Process: Reflections on Invention in the Arts and Sciences* (Berkeley, CA: University of California Press).
[10] Perkins, D., 2000, *Archimedes' Bathtub: The Art of Breakthrough Thinking* (New York: W.W. Norton and Company).
[11] Harris, T., 2000, *Hannibal* (New York: Dell Publishing).
[12] Di Martino, P. and Zan, R., 2001, Attitude toward mathematics: Some theoretical issues. *Proceedings of 25th Annual Conference for the Psychology of Mathematics Education*, Vol. 3, pp. 351–358.
[13] McLeod, D., 1992, Research on the affect in mathematics education: A reconceptualization. In: D.A. Grouws (Ed.) *Handbook of Research on Mathematics Teaching and Learning* (New York: Macmillan), pp. 575–596.
[14] Eynde, P., De Corte, E. and Verschaffel, L., 2001, Problem solving in the mathematics classroom: A socio-constructivist account of the role of students' emotions. *Proceedings of 25th Annual Conference for the Psychology of Mathematics Education*, Vol. 4, pp. 25–32.
[15] Ponte, J., Matos, J., Guimarães, H., Cunha Leal, L. and Canavarro, A., 1992, Students' views and attitudes towards mathematics teaching and learning: A case study of curriculum experience. *Proceedings of 16th Annual Conference for the Psychology of Mathematics Education*, Vol. 2, pp. 218–225.
[16] Ajzen, I., 1988, *Attitudes, Personality, and Behaviour* (Milton Keynes: Open University Press).
[17] DeBellis, V. and Goldin, G., 1993, Analysis of interaction between cognition and affect in elementary school children during problem solving. *Proceedings of 15th Annual Conference for the Psychology of Mathematics Education – North America Chapter*, Vol. 2, pp. 56–62.
[18] DeBellis, V. and Goldin, G., 1999, Aspects of affect: Mathematical intimacy, mathematical integrity. *Proceedings of 23rd Annual Conference for the Psychology of Mathematics Education*, Vol. 2, pp. 249–256.
[19] Leder, G., 1992, Measuring attitudes to mathematics. *Proceedings of 16th Annual Conference for the Psychology of Mathematics Education*, Vol. 3, pp. 33–39.

[20] Liljedahl, P., 2004, The AHA! experience: Mathematical contexts, pedagogical implications. Unpublished doctoral dissertation, Simon Fraser University, Burnaby, British Columbia, Canada.
[21] Liljedahl, P., 2004, Mathematical discovery: Hadamard resurrected. *Proceedings of 28th Annual Conference for the Psychology of Mathematics Education*, Vol. 3, pp. 249–256.
[22] Hofstadter, D., 1996, Discovery and dissection of a geometric gem. In: J. King and D. Schattschneider (Eds) *Geometry Turned On: Dynamic Software in Learning, Teaching, and Research* (Washington, DC: The Mathematical Association of America), pp. 3–14.
[23] Liljedahl, P., 2004, Improving problem-solving journals: The mathematician, the narrator, and the participant. *Proceedings of 26th Annual Conference for the Psychology of Mathematics Education – North America Chapter*, pp. 439–445.