Devlin's Angle

May 2008

Lockhart's Lament - The Sequel

In last month's column I discussed a classic calculus problem often referred to as the "napkin ring problem." Although it appears at first glance like any one of dozens of volumes or revolution problems that calculus instructors give their students to practice their mastery of integration, this particular problem has a surprising answer. The volume of the napkin ring does not depend on the radius of the sphere from which a cylinder is removed to create the ring, but only on the height of the cylinder.

The proof I gave at the time was (deliberately) the "by the book", pedestrian one. There is nothing difficult about it, and it does provide a perfectly good exercise in integration. Any student who can carry out the calculation I gave has demonstrated mastery of the technique for calculating a volume of revolution. No ingenuity is required. It's a routine application of integration. The question is, what is the student's response on seeing that surprising answer? Mathematics teacher Paul Lockhart would hope - wish - that the student would be prompted to ask "Why?" and would then seek to find an explanation. (Yes, last month's column was a set-up. I must have watched too many episodes of Prison Break.)

We met Lockhart in my March column, which was devoted to publication, for the first time, of an essay he had written back in 2002. In that essay, Lockhart argued for teaching that awakened and stimulated students' natural curiosity. We'll come to that argument momentarily. For the moment, let me see if we can do as Lockhart would hope and figure out just what is going on with the napkin ring.

Once you have solved the problem the pedestrian way and found the formula for the volume of the napkin ring, it doesn't take most mathematically able individuals long to come up with another derivation. What is more, provided you know the formula for the volume of a sphere, that alternative derivation makes no use of calculus at all!

If you have not come across the napkin ring problem before, you might like to try to solve it for yourself without using calculus. Otherwise, you will find the (non-calculus) solution here.

Reflections on Lockhart

Now back to Lockhart. As I had suspected (and hoped), the appearance of Paul's essay generated a massive response, some of it coming to me, the bulk going directly to Paul himself. The remainder of this month's column is devoted to a summary of some of the emails we received, with some editorial comment from me and a lengthy response by Paul.

By far the greater portion of the emails I received, and I gather almost all the ones Paul received, were largely congratulatory or else highly favorable to the essay. My editorial focus here is, however, on the responses that took issue with one or more of the points he made. For the writers made what I believe to be valid criticisms.

[BTW, it never ceases to astonish me that some readers assume the job of an editor or a columnist is to write or give voice only to articles or opinions that he or she agrees totally with. When I published Paul's essay, I wrote: "It is, quite frankly, one of the best critiques of current K-12 mathematics education I have ever seen." That doesn't mean I agree with everything in the essay, for heaven's sake! In fact, it doesn't imply that I agree with *anything* he said, though as it happens I agree with many of his points. Lockhart wrote eloquently and with passion about an issue he is intimately familiar with, raising many important points. And as someone whose career has included both high level mathematical research and K-12 mathematics teaching, he brings a perspective that relatively few MAA members can claim. That's why I wanted to bring his essay to a wider audience.]

One question raised in my mind on reading Paul's essay was, what can we learn in terms of our mathematics education *system*? Leaving aside for a moment the pluses and minuses of the kind of approach he advocates, is it reasonable to expect that we could provide all school pupils with a similar experience? I fear we all know the answer to that one. Heavens, pupils in the K-12 system are lucky if they are taught mathematics by someone who has taken more than one or two college courses in math, let alone majored in the subject. And to teach Paul's way requires (I believe) far more than having a bachelors degree in mathematics. It requires someone very much like himself, someone who loves mathematics and has mastered it to a professional level. (A successful research career, as Paul has under his belt, is probably overkill, though I am sure it helps in a number of ways.)

Though I would love to see every school student exposed to real mathematical thinking and stimulated the way Paul advocates, I think it is simply not feasible. (Not that Paul claims it is; as he points out in his response below, his essay is a lament, not a proposal.) It is, I suggest, inescapable that at the systemic level, we cannot avoid having to provide classroom teachers with a fairly well-specified prescription to follow, and we must accept that many of them will be unable to deviate much, if at all, from that prescription. This does not mean there is nothing to learn from Paul's experience in terms of curriculum specification. A fairly thorough prescription should not shackle teachers. It would be a tragedy if the system prevented talented instructors from providing their pupils with the kind of stimulating experiences Paul describes. *A Mathematician's Lament* may not be a proposal, but we can surely learn a lot from what Lockhart says.

Another reaction I had to the essay was that Paul's approach is geared to developing in his pupils a love for mathematics as an enjoyable and challenging intellectual pursuit. Now, there is no doubt that for many of us, mathematics is precisely that. Nothing wrong with trying to foster it in as many young minds as we can. But mathematics has another face. It is one of the most influential and successful cognitive technologies the world has ever seen. Tens of thousands of professionals the world over use mathematics every day, in science, engineering, business, commerce, and so on. They are good at it, but their main interest is in its use, not its internal workings. For them, mathematics is a tool. Even if they had an interest in investigating the inner workings of that tool (and there are plenty who claim they do not), they do not have the time; the problems they are trying to solve are simply too pressing and too demanding. Having solved the napkin ring problem the routine, pedestrian way, by integration, for them the issue is over - the problem is solved - and it's time to move on. This is the utilitarian face of mathematics I talked about in my January and February columns, in connection with national competitiveness in the global economy. I fear that Paul's approach would not serve those individuals particularly well. Exposure for an initial few years, perhaps yes, and maybe a term or two thereafter. But as a nation I don't think we can afford to take it as the norm. Industry needs few employees who understand what a derivative or an integral are, but it needs many people who can solve a differential equation.

As you will see, Paul himself counters this by drawing a distinction between K-12 education and college level. Personally, I'm not convinced that it will work to leave the "training for a competitive economy" stuff to college level, but it sure would be nice - and I believe would help us remain a competitive economy - if students were exposed to the kind of experience Paul advocates *throughout* their mathematics education, if such were possible.

I had other thoughts too, as I read, and then re-read Paul's lament. But most of them were also addressed by the emails I received, so I'll let those respondents speak for themselves. Then I'll let Paul respond.

What you said about Lockhart's lament

One reader (with a Ph.D. in mathematics) sent me an email that seemed to encompass much of my own thinking:

"I was very impressed with "Lockhart's Lament" that you recently posted on Devlin's Angle. Lockhart presents one of the most uncompromising versions I have encountered of the hedonistic approach to mathematics education: unless learning mathematics is fun it's no damn good. This is a position for which I have a lot of sympathy. As a kid I was drawn to mathematics precisely by its fun aspects, emphatically not by its utility. But while the hedonistic approach is probably feasible at the level of the classroom, and perhaps even the school, the difficulty comes when we try to scale up to the level of a "system," such as a county, state, or nation. For educational systems almost inevitably entail measuring results, an activity from which Lockhart clearly recoils. "There should be no standards, and no curriculum. Just individuals doing what they think best for their students." (p. 23) Furthermore, mathematics is in fact useful, and Lockhart surely goes overboard in relegating this utility to such a secondary position. I infer that he thinks the useful aspects of mathematics can be picked up as needed at any time, by anyone imbued with the true spirit of mathematics. Perhaps, but does not the hedonistic approach, taken to its extreme, risk producing students only willing to tackle problems that please their aesthetic sense? And it is alas true that mathematics can be usefully applied by many who possess little or no appreciation for its beauty. This being so, why should society expend resources to impart knowledge of this beauty? One might argue that aesthetic appreciation of mathematics in some way makes a person better at applying mathematics in even the most mundane setting. I'm not sure this is true, and have no idea how one could ever compile evidence to support it. It would pose a special challenge to Lockhart, with his aversion to measuring educational results. He is reduced to something like, "Trust me, this is the best way to teach math for all purposes."

Such are the pragmatic objections to Lockhart that occur to me. Of course Lockhart is no pragmatist, proudly so. But as a goad to rethinking the most basic issues of math education, as a bomb for exploding conventional notions, his essay is valuable indeed."

Another reader wrote to me:

"Paul Lockhart is right that there is much that could be done to improve mathematics education. He is right that the current curriculum contains too much material and is too heavy on facts and skills that are easy to test. But his idea that mathematics is a pure art form that should be appreciated and taught as such is wrong. His ideas about art are equally wrong. The idea of art as free expression is romantic folly. Artists are problem-solvers. They are working for a living. They produce, play music, and dance for others for money. My artist friends, the successful ones, are taken up much of the time considering how to perfect or extend their craft and how to sell more product.

I studied graphic arts and designed books, fliers and posters. While I was taking a design course [my instructor] passed by me as I was laboring over a design project for him. "Have you solved it yet?" he asked. That was when I realized that the essence of art was applied problem-solving... [Let me point out how] completely erroneous [many popular] ideas about success in the arts are: as if one somehow either was born with the ability to play the violin or not. Talent plays a role, but time-on-task is the great *determiner of achievement in playing an instrument and in doing mathematics. These arts are mastered at the cost of sweat, and their practice is not easy.*

If the point is that more time should be available to develop mathematical ideas in class and that teachers should be under less pressure to cover techniques and should have more time to explore ideas with students, then I am already with you. But the sort of arguments that Paul Lockhart has in this piece do not advance that cause. Moreover, since they consign everyone (educators, the public, teachers, and publishers) to perdition, this seems to leave Paul Lockhart alone to do the good thing. This will not work."

Another reader (a university professor) took Paul to task for ignoring the history behind the current mathematics curriculum:

"As I see it, Paul Lockhart's essay would be much more powerful if it were not written in such a complete historical vacuum. Although Lockhart decries the sterile formalism in which mathematics courses have been and continue to be taught, he makes absolutely no reference to the fact that the traditional mathematics curriculum was demolished by the excessive formalism and abstractions of the SMSG new math, as incorporated in the Houghton Mifflin series of books co-authored by Mary P. Dolciani. This apparent ignorance on Lockhart's part is likely due to the fact that he was educated with Dolciani-type books, and he may not be aware of the preceding textbooks."

Finally, another e-mailer (also a university professor) quoted the following passage from Lockhart's lament:

"All metaphor aside, geometry class is by far the most mentally and emotionally destructive component of the entire K-12 mathematics curriculum. Other math courses may hide the beautiful bird, or put it in a cage, but in geometry class it is openly and cruelly tortured. (Apparently I am incapable of putting all metaphor aside.)"

The e-mailer then went on to write:

"I am not alone among Ph.D.'s in mathematics for whom traditional proof-based Euclidean geometry, T-proof even with its idiocies, was the first real introduction to a lifetime of mathematics. And communication of geometry, proof-based geometry, remains among my favorite topics although none of my professional work is in geometry. The subject is beautiful and the logic, semi-formal deductive reasoning, remains the same - and hugely important to human thought - over millennia. The fact that it has been eliminated - or distorted beyond reason - from the precollegiate preparation of many strong college-bound students borders on sinful."

Although this passage was part of a message that was extremely critical of Lockhart's essay, I'm not entirely sure that on this point the two are fundamentally at odds in principle, though I'm definitely with Lockhart in being opposed to "formal proof" as so often practiced in school classrooms, whereas I gather the email writer sees benefit (presumably a net benefit) in what are sometimes called cookie-cutter proofs.

Lockhart responds

First off, let me take this opportunity to thank Keith for offering to publish my essay, and to thank all of you who have written in with your comments and questions. The response has been absolutely wonderful.

I would like to begin by reminding readers that what I have written is a Lament, not a Proposal. I am not advocating any particular plan of action; I am merely describing the extremely sad and painful (and probably hopeless) state of affairs as I see it: mathematicians are not interested in teaching children, and teachers are not interested in doing mathematics.

If I am advocating anything, it is only the obvious (and time-tested) idea of "learning by doing." If I have a method, it is only to convey my love for my subject honestly, and to help inspire my students to engage in a delightful and fascinating adventure - to actually do mathematics, and to thereby gain an appreciation for the depth, subtlety, and yes, utility, of this quintessentially human activity. Is that really such a strange and radical idea? Have we really reached a point where one has to argue for teaching that "awakens and stimulates students' natural curiosity?" As opposed to what? I thought that was the definition of teaching!

I find it a bit frustrating that I am put in the position of having to defend such a simple and natural idea as having students engage in the actual practice of mathematics. Shouldn't it rather be the proponents of the current regime who should have to defend their bizarre system, and explain why they have chosen to eliminate from the classroom the actual ideas of the subject? You say I take a hedonistic approach to mathematics education? I call it a mathematical approach to mathematics education!

What I find so pathetic about our math education system is that it reduces a lively, creative, and messy human art form to a sterile set of notations and procedures, then attempts to train students to master them and become "technically skilled." Of course it fails even on its own terms because there is no coherent narrative - the teacher doesn't know where the natural logarithm came from, what its problem history is, what it means within the context of modern mathematics, only that it's on the test and the students need to "know" it. So the students cram some formulas into their heads for a day or two, pass a test, and promptly forget them. Of course most people can't retain dry, meaningless hieroglyphic information that they had no role in creating or contextualizing, so they get classified by the teacher (and by themselves) as "bad at math." (I worry that the most talented mathematician of our time may be a waitress in Tulsa, Oklahoma who considers herself bad at math.)

What are the goals of K-12 mathematics education?

One theme that seems to recur in discussions of my essay is this idea of training the 21st century workforce to be responsive to the needs of industry and to be "competitive in the global economy." I am no economist, but this seems to be more a matter concerning college and graduate level education, not the K-12 setting with which my essay is nominally concerned. Of course (as you may easily imagine) I have quite a bit to say about the disastrous state of affairs at the university level, but perhaps this deserves a separate discussion. (I have, however, received numerous emails from graduate students and researchers in mathematics and the physical sciences who feel that my essay hit the nail on the head for them as well.) So let's save the economic discussion for another time.

So the question is, what should be the goals of K-12 mathematics education? Or, to put it in somewhat more inflammatory terms, what whole categories of human experience do you want hidden from your child? Any other "enjoyable and challenging intellectual pursuits" you wish to prevent your youngster from engaging in? Painting and music certainly don't seem very practical, and neither does all this literature and poetry. Why should society expend resources to impart knowledge of any form of beauty? My god, there's so much unprofitable, non-industrial fluff our young economic units are being wastefully exposed to!

But seriously, are we really saying that introducing children to mathematics and helping them to develop a mathematical aesthetic is a bad thing? Inspiration, wonder and excitement can only lead to positive results. And it is especially valuable to have this kind of energy and enthusiasm when learning to master a new technical skill. Practicing a new scale is a lot easier when it occurs as part of an interesting, challenging, and beautiful piece of music.

Look. A child will have only one real teacher in her life: herself! I see my role as not to train, but to inspire and to expose my students to a wide range of ideas and possibilities; to open up new windows. It is up to each of us to be students - to have zeal and interest, to practice, and to set and reach our own personal artistic and scientific goals. Children already know how to learn: you play around and have fun and struggle and figure it out for yourself. Grownups don't need to hold infants up and move their legs for them to teach them to walk; kids walk when there is something interesting in the room that they want to get to. So a good teacher is someone who "puts interesting things in the room," so to speak.

No? Alright, fine. I propose a curriculum for reading which has students first learn all the words that begin with the letter 'A' and then proceeds through the alphabet. The course of study would be divided into 26 Units, and naturally one could not 'skip' to the advanced 'Q' class without having taken the 'P' prerequisite. (Reading actual books would come much, much later of course.) I wonder why we don't currently do this? Could it be because parents and teachers actually do read from time to time, so they know what matters and what does not? But the only source of information about what mathematics actually is comes from school itself: the 37th-generation photocopy of the same blinkered misconceptions, the perpetual feedback loop of School Math.

Suppose the devil were to offer you this deal: your child will get a perfect score on the English section of the SAT, but will never again read a book for pleasure. I would like to believe that no parent would make that deal. But how many would gladly shake the devil's other hand? Math is not something we want our children to enjoy, it is something we want them to get through.

Pure math or applied?

Another thing that strikes me is how often I am placed on the wrong side of some sort of Pure vs. Applied, or Art vs. Technology debate. I have always found these to be false dichotomies. Mathematics is an incredibly rich and diverse subject. Can't we enjoy it in all of its many shades and textures? Besides, what could possibly be more useful than a lifetime of free entertainment?

As Salviati says, just because I object to a pendulum being too far on one side doesn't mean I want it to be all the way on the other side. I seek a balance. Can we not have Theory and Practice, Beauty and Utility? I may be a so-called "pure" mathematician, but that doesn't prevent me from enjoying electronics and carpentry (and oil painting too, by the way). And yes, artists are problem-solvers. And problem solving is an art! By the way, many people (such as myself) enjoy drawing and painting and playing music for fun; not all artists are in it for the money.

The Pure/Applied distinction is one that I loathe. It is the creative/mindless distinction that I care about. Whether you are proving an abstract theorem about group schemes or calculating an approximate solution to a differential equation, you are either being a creative human being pursuing your curiosity or you are mindlessly following a recipe you neither understand nor care about. That's the issue for me. And if all you are interested in is having a rote mechanical algorithm performed quickly and accurately, isn't that what we build machines for?

My point is that at present we have neither Romance nor Practicality - nothing but a jumbled, distorted mishmash of pseudo-mathematical vocabulary, symbols, and mindless procedures. It is as if some extraterrestrial Captain Cook accidentally left behind a protractor and a logarithm table, and School Math is the "cargo cult" the natives have reconstructed. The current utilitarian regime is a complete failure. Not only do students have no idea of what the subject is actually about, they can't even remember any of this supposedly "useful" information from one week to the next. That's what happens when you remove the coherent narrative; and of course that's what happens when you remove the students from the creative process.

Finally, in response to my mathematician friend, of course I have no objection to formalism per se. In fact, quite the contrary. The advent of formalism in mathematics is a crucial (and beautiful!) development. But there is a huge difference between a group of professional mathematician-philosophers (e.g. Euclid, Weierstrass) attempting to formalize and axiomatize the current state of their subject, and a roomful of fourteen-year-olds thinking seriously about shapes for the first time in their lives. My complaint (about High School geometry) is that the formalism is being attempted too soon, and not by the students. And I make no mention of the reasons or history behind the current disaster because I frankly don't care which committee of idiots did what when; I care about mathematics and children.

And I certainly do care about measuring educational results. But what is an "educational result?" The twinkling eyes of my students, together with their heartfelt and beautifully expressed mathematical arguments are all the results I need.

To be continued ...

Not necessarily in my column, but throughout our profession, I hope. Thank you, Paul. And thanks to the many of you who wrote to one or both of us. KD.

Devlin's Angle is updated at the beginning of each month.

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