# MATHEMATICAL MODELLING: FROM NOVICE TO EXPERT

by

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# Abstract

This study strives to understand how mathematical modelling is perceived by novice, intermediate and expert modellers, through comparing and contrasting their understanding and habits of modelling. The study adopted a qualitative methodology based on observations, interviews and surveys of 78 participants. This included 14 experts who are professors, 11 intermediates consisting of graduate students and post-doctoral fellows, and 53 undergraduates or novices. The study incorporated interviews of the professors and the post-graduate participants, while questionnaires were utilized to understand the perspective of the undergraduate students. The study revealed that the majority of expert participants see modelling as a collaborative effort. There is a dichotomy among them regarding whether mathematical modelling is the setting up of a mathematical model alone, which is deemed an art, or if it includes the solving of the model, which is more a science. These differences have implications on how modelling is taught and how novices and intermediates in turn will view the modelling process. Experts also vary in their opinion on whether models must be verifiable or not. One key feature of the experts approach is that they begin by assuming that they do not understand the question asked and work to ensure that they do. This is despite their superior ability to solve problems. Intermediate participants were more forthcoming with their emotions on modelling than experts; they cited research as opposed to collaboration as their primary means of dealing with barriers arising during the modelling process, and gave credit to intuition as a skill needed for solving - something not mentioned among the experts. Novices were the most descriptive about their feelings when modelling. They conveyed a tendency to be more passive when encountering barriers, waiting for help or giving up as opposed to actively working through the problems. Many of our results, including those mentioned above, have implications for the teaching of effective mathematical modelling.

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# Contents

# List of Tables

# Chapter 1

# Introduction

This story begins in 2007 when I first decided to model crowd flow as a “thinking” fluid. I had done some work on this in my Computational Fluid Dynamics class and was pursuing it further as my thesis topic. I found the work difficult to say the least. All Ph.D. work is probably daunting at the beginning, but I was completely overwhelmed. I had no idea what modelling was! How to start? What factors to include? What to do? I was constantly stuck and unable to get unstuck without the specific and constant help of my supervisor. My background had had little or no “real” modelling in it. Yes, I had worked on many equations that modelled situations in real life during my master’s degree, but I had never started “from scratch”.

After working on crowd flow for a year with little progress, and reaching a point where I was considering quitting my Ph.D. completely, I entered into a discussion with my supervisor. He is an excellent modeller himself and teaches courses at SFU on modelling. He had attended and hosted several modelling camps and workshops and understood how to work collaboratively with others to acquire desired results. I asked him what I thought then were simple questions: How does one DO modelling well? What are the skills needed to be a good modeller? I thought that perhaps if I could work on these skills then I too could figure out how to do my then thesis work. This last line of questioning became the starting point of an even greater discussion, as we could not pinpoint “what the magic was”.

I thought perhaps it was difficult for him to articulate exactly what his process was, as it had become instinctive after doing it for so long. So I tried to tackle this from a different perspective, asking him what his learning objectives were when teaching a modelling class. This yielded another interesting discussion, this time about identifying modelling outcomes.

This was significant as many outcomes were not simply the particular mathematical content he was covering. How does one know at the end of a modelling class that students have learned the skills required to do modelling well? How can we check that the students who do well are not simply those that had already acquired the necessary mathematical modelling skills before taking the class? In other words, if the final exam was given on the first day of class, apart from their understanding of the content knowledge, would they do the same as at the end of the course?

This led me to ask what I then considered the most basic question of all: What IS modelling? Even this proved to be unclear since definitions varied in textbooks and modelling literature alike. Modelling is a broad area and to try to define it is difficult. And so began my quest and a new thesis topic to understand this aspect of mathematics which I had little experience with, but realized was a very important part of doing applied mathematics.

## 1.1 Motivation

Mathematical modelling is an important aspect of the applied mathematics curriculum. It provides students, particularly graduate students, with the skills to succeed professionally in industry. It gives these students the tools to analyze, understand and forecast based on data that in this age is easily accessible to them. Modelling helps students to transfer the knowledge that they have learnt in their less open-ended classes, to real-world problems. This transfer of knowledge is a skill that students can go on to use in the workplace and other areas of study. Since most mathematics and applied mathematics students will not go on to be mathematicians, these skills had better prepare them for whatever they aspire to do next. In the 1970's McLone [1] reported that mathematics graduates had difficulty when moving from the classroom to the workplace:

> "Good at solving problems, not so good at formulating them, the graduate has a reasonable knowledge of mathematical literature and technique; he has some ingenuity and is capable of seeking out further knowledge. On the other hand the graduate is not particularly good at planning his work, nor at making a critical evaluation of it when completed; and in any event he has to keep his work to himself as he has apparently little idea of how to communicate it to others." ( [1], p.33)

In 1995 this was still the case and PhD students reported not feeling able to tackle real world problems in industry [2].

As many universities promise to prepare their students to engage the world, modelling is one area that allows universities to fulfill the promise of "equipping students with the knowledge, skills, and experiences that prepare them for life in an ever-changing and challenging world" [3]. To this end universities often have courses, undergraduate degrees and graduate degrees that cater to the modelling aspect of applied mathematics. This is precisely because mathematical modelling links mathematics to real life phenomena. "It occupies a middle ground between mathematics and most other sciences and engineering disciplines" [4].

In order to analyze the entire world around us, mathematical modelling is key. Modelling problems are found in "physics, engineering, chemistry, computer science, biology and even in such subjects as psychology and sociology" [4]. Mathematical models "provide insight into how various forces act to change a cell, an organism, a population, or an assemblage of species" [5]. They help to address the complex questions arising in healthcare where simple intuition is not enough, allowing us to "develop solid, defendable, evidence-based answers to those questions" [6]. This is just the tip of the iceberg.

Industrialists also value mathematical modelling, as it is cheaper than doing full experiments or simulations in many cases. Many modelling problems come from industry, where the detailed mechanics of the problems are often not as well known as one might think due to the expense of experimentation: "when you see the operating conditions – ferocious temperatures, inaccessible or minute machinery, corrosive chemicals – you realise how expensive and difficult it would be to carry out detailed experimental investigations" [7]. Not only is it difficult and expensive to use experiments to understand some of the industrial problems, but sometimes it is simply impossible. The use of mathematical models allows us to explore the problem more extensively:

> "It is often easy to vary parameters in the mathematical model over wide ranges, whereas this may be very time-consuming or expensive, if not impossible, in an experimental setting." ( [8], p.50)

Mathematical modelling is clearly an important skill for mathematics students to learn as well as those from other disciplines. Having identified the significance of modelling to universities, students, industrialists and those in other disciplines, the next step is to identify the areas of concern when doing and teaching modelling.

## 1.2 The Problem

Mathematical modelling courses have several problems that a classic mathematics course (such as an introductory differential calculus course) would not have. One problem with teaching mathematical modelling is that we want to use precise mathematics to fit imprecise problems for which there may be no well-defined solution at all!:

> "It is the nature of real-world problems that they are large, messy and often rather vaguely stated. It is very rarely worth anybody's while to produce a 'complete solution' to a problem which is complicated and whose desired outcome is not necessarily well specified (to a mathematician). Mathematicians are usually most effective in analysing a relatively small 'clean' subproblem for which more broad-brush approaches run into difficulty." ( [7], p.4).

In many areas of mathematics the information needed is given and clearly laid out. However, in modelling oftentimes much of the information must be discarded in order to boil the problem down to its essence. This is non-rigorous, and requires creativity. A subsequent issue that arises is what the learning outcomes should be in a mathematical modelling course or degree. How should the instructor of modelling approach the subject, and what aspects should she focus on?

> "The preferred mode of "teaching" modeling is quite problematic: Does one have the students gain experience by having them face real problems (usually ill-defined) and thus learn by doing, or does one use a book or books presenting them with worked-out approaches? Should they know a fair amount of math first, or should they learn what is necessary as they face new challenges? Does one mix the areas of problems, the continuous or discrete setting, the level of mathematical sophistication? There is a host of hypotheticals.
>
> I have given modeling courses quite a number of times, but I don't have many strong views on the answers to what I have posed above. I have varied what I have done and never been completely satisfied. (And establishing a student grade is quite a challenge!)" [9].

The problem is made worse by the fact that some mathematics students believe from elementary school onwards that mathematics problems all have a unique closed-form solution that the teacher knows – the proverbial "right answer." This is to be expected on a developmental level as college students begin with a belief that there are right and wrong answers that can be acquired from the authorities and are to be memorized, as explained by Perry in his work on intellectual development [10, 11]. However, this belief is in contrast with the way mathematical modelling works as we will see in the literature discussed in the following chapter (Chapter 2).

## 1.3 Summary

This introduction illustrates that mathematical modelling is an area of importance in which experts do well. However, due to the messy nature of the problems and the issues involved when attempting to teach modelling, novices have difficulty with it. The practices of novices are significantly different from those of the experts, and novices are more inclined to be overwhelmed by the enormity of the task of modelling [12,13]. The literature addresses some of the issues that novices have but additional work must to be done in order to understand the transition from novice to expert in more detail [14]. In my effort to understand the true nature of mathematical modelling, I first attempted to clarify what modelling means as it covers such a broad range of skills and topics. I also looked to identify some of the issues associated with the transition from novice modeller to expert. This was accomplished by working with modellers with a range of ability to identify the differences in the skill-set, behaviour, and attitude of the varying groups. In particular, I tried to understand what skills the experts use to move from being stuck to becoming unstuck.

In Chapter 2, I discuss the literature pertaining to modelling including the transition from novice to expert. This discussion highlights that there are still some deficits in the modelling literature. A review of problem solving literature as well as literature on the proposed psychological contributions suggest a framework for analysing the research questions that emerge from the literature review. To answer those aforementioned research questions, I describe the methods employed for conducting this study, in Chapter 3. This includes a discussion of the participants who range from novice to expert, alongside an explanation of the data-gathering process. I also identify the limits of the chosen research methodology. In Chapters 4 – 7, I present the results and analysis of the study, and discuss conclusions in Chapter 8. Finally, in Chapter 9, I suggest some implications these results have on the teaching of mathematical modelling at the tertiary level and directions for future work.

# Chapter 2

# Literature Review

Mathematical modelling has a short but rich history in the literature, which highlights several of the issues of the mathematical modelling process. I studied the mathematical modelling literature, focusing principally on modelling textbooks, as these are the primary resource for teaching and learning modelling. I also looked at articles found in the International Community of Teachers of Mathematical Modelling and Applications (Ictma) journals, that discuss differences between the novice and the expert modeller. This was followed by a look at the problem-solving literature, highlighting the similarities between problem-solving and modelling while attempting to gain some insight into the issues that the modelling literature has left uncovered. Finally, I investigated the possible psychological issues that may affect the modelling process.

## 2.1 Mathematical Modelling Literature

In order to start looking at the broad issue of what exactly mathematical modelling is, I examined the mathematical modelling literature to begin addressing the questions introduced in Chapter 1. I looked at two different types of literature on modelling: textbooks and Ictma journal articles. These two types of literature encompass two different perspectives: those who do and teach modelling and those who research modelling education and culture. I began with a look at mathematical modelling textbooks, as these are the primary tools for teaching modelling and are often a student's first introduction to the field.

There is a wide range of mathematical textbooks available, but I focused on two categories: textbooks written by mathematicians primarily for students of mathematics and textbooks written by scientists in different fields to cater to students in various scientific disciplines. I felt that the varying backgrounds of both the authors and the intended end users would provide a spectrum of approaches to teaching modelling.

In the first textbook category, those written by mathematicians primarily for students of mathematics, I examined four books: *Mathematical Modelling: A Case Studies Approach*, by Illner *et al.* [4]; *Practical Applied Mathematics: Modelling, Analysis, Approximation*, by Howison [7]; *Mathematical Models in the Applied Sciences*, by Fowler [15]; and *Ants, Bikes and Clocks: Problem Solving for Undergraduates*, by Briggs [16]. Note here that although the title of Briggs' textbook includes "problem-solving", he specifies in the introduction of his work that "this book highlights modelling or story problems" ( [16], p.2).

In the second category of modelling textbooks, those written by scientists in different fields to cater to students in various scientific disciplines, I examined three books: *The Nature of Mathematical Modeling*, by Gershenfeld [17] (Physicist and Computing Scientist), *A Biologist's Guide to Mathematical Modeling in Ecology and Evolution*, by Otto (Zoologist) & Day (Mathematical Biologist) [5], and *Modelling in Healthcare*, by SFU's Complex Systems Modelling Group or CSMG [6] (Mathematicians, Physicists, Epidemiologists). These two categories of authors approach the definition and the teaching of modelling in different ways.

### 2.1.1 Issues with Modelling: Definition

The first thing of note was the variation in definition of mathematical modelling. Illner *et al.* give a very broad definition of modelling:

> "Mathematical modelling is a subject without boundaries in every conceivable sense. Wherever mathematics is applied to another science or sector of life, the modelling process enters in a conscious or subconscious way." ( [4], p.xi)

This definition certainly covers the breadth of modelling but is not a working definition. Consistent with this, Howison explains that modelling should not be precisely defined:

> "There is no point in trying to be too precise in defining the term mathematical model: we all understand that it is some kind of mathematical statement about a problem originally posed in non-mathematical terms." ( [7], p.4).

Howison goes on to explain that while all models do not explain cause and effect, all useful models should be predictive. Illner *et al.* make no such claims; however, Howison's colleague Fowler also raises the issue of predictive power in modelling:

> "A model is a representation of a process. Usually, a mathematical model takes the form of a set of equations describing a number of variables [...] Applied mathematicians have a procedure, almost a philosophy, that they apply when building models. First there is a phenomenon of interest that one wants to describe or, more importantly, explain. Observations of the phenomenon lead, sometimes after a great deal of effort, to a hypothetical mechanism that can explain the phenomenon. The purpose of a model is then to formulate a description of the mechanism in quantitative terms, and the analysis of the resulting model leads to results that can be tested against the observations. Ideally the model also leads to predictions, which, if verified, lend authenticity to the model. It is important to realize that all models are idealizations and are limited in their applicability. In fact one usually aims to over-simplify" ( [15], p.3).

Fowler's definition of modelling is initially quite succinct, but he goes on to elaborate with a description of a modelling cycle, which is the applied mathematician's philosophy of building a model [15]. He acknowledges that going from the observation to the mechanism sometimes requires significant effort, and also raises the idea of simplification often being the aim. Fowler also describes some of the issues that are associated with mathematical modelling, in particular the teaching of it:

> "Mathematical modeling is a subject that is difficult to teach. It is what applied mathematics (or to be precise, physical applied mathematics) is all about, and yet there are few texts that approach the subject in a serious way. Partly, this is because one learns it by practice: There are no set rules, and an understanding of the 'right' way to model can only be reached by familiarity with a wealth of examples."( [15], p.3).

Fowler's view that modelling is learned by practice is a very common one (see Gershenfeld and Otto & Day below). He also implies by his use of quotation marks that the right way to model is somewhat subjective.

These three definitions show an increase in specificity. Briggs' definition of modelling appears along the same lines at first glance:

> "Modeling or story problems are quantitative problems that are posed in a realistic context. A key distinction of these problems is that they are not posed explicitly as mathematical problems. For this reason, their solution requires an essential preliminary step that may be the crux of the solution. That step, often called *modeling*, is to transform the stated problem from words into mathematics. Having formulated the problem in mathematical terms, it must still be solved!" ( [16], p.2).

On the surface this seems to be re-iterating Fowler's description of going from observation to mechanism, description in quantitative terms, followed by analysis; however, it is qualitatively different. Briggs identifies modelling as the step from phenomenon to mathematics only. What follows after is no longer modelling, but solving. Briggs also does not require the question to actually come from reality, only that it be posed in a realistic context, and makes no mention of predictive ability. Therefore it cannot be construed that mathematicians who do modelling all have the same working definition.

Interestingly, neither Gershenfeld nor Otto & Day explicitly define what a mathematical model is. They describe issues or steps involved when building a model, but never specifically say what it is that they are building. Gershenfeld does not even describe a modelling cycle but instead raises some issues involved in the modelling process:

> "To build a model there are many decisions that must be made, either explicitly or more often, implicitly. Some of these are shown in Figure 1.1. Each of these is a continuum rather than a discrete choice. This list is not exhaustive, but it's important to keep returning to it: many efforts fail because of an unintentional attempt to describe either too much or too little.
>
> These are meta-modeling questions. There are no rigorous ways to make these choices, but once they've been decided there are rigorous ways to use them. There's no single definition of a "best" model, although quasi-religious wars are fought over the question." ( [17], pp.1-2)

Gershenfeld's explanation of modelling decisions being made implicitly and without rigour hints at the lack of precision in the modelling process. He also raises the idea of there being no “best” model, which is tied to Fowler’s statement of “the ‘right’ way to model.”

Otto & Day also avoid defining mathematical modelling but raise several issues involved in doing modelling. These issues touch more on the feelings that might be experienced when modelling:

> “If you have seen mathematical models but never constructed one, it may appear like an overwhelming task. Where do you start? What is the goal? How do you know whether the model makes sense? This chapter outlines the typical process of modeling and gives helpful hints and suggestions to break down the overwhelming task into manageable bits. The most important piece of advice is to start. Start thinking about problems that puzzle you. Grab a piece of paper and start drawing a flow diagram illustrating various processes at work. The biggest hurdle preventing most biologists from modeling is the paralysis one feels in the face of mathematics; [...] Over time, you will learn more tools and techniques that will allow you to avoid pitfalls and to get further with the problems that interest you. Your intuition will develop to help you “see” when something is wrong with your model and to help you interpret your results” ( [5], p.17)

Otto & Day not only acknowledge the difficulties of modelling, but also suggest how to deal with them (just start) and reassure the reader that in time their intuition will develop. They go on to describe a seven-step modelling process (see Table 2.1), again acknowledging some student concerns:

> “Box 2.1 describes, in seven steps, how to construct a dynamical model. This is like describing how to ride a bike in a series of steps; obviously we can only give an idea about how the process works. Mastering the steps requires practice, [...] the first step, coming up with a question, can be more difficult than it sounds. In most biology classes, students are told what the questions are and what answers have been found. Rarely are students asked to formulate scientific questions for themselves. This is very unfortunate because, in any scientific enterprise (modeling or otherwise), the process begins with a question.” ( [5], pp.17-19)

Otto & Day address the fact that the initial step in modelling (coming up with a question) is a difficult one. This reinforces both Fowler’s statement that moving from phenomenon to hypothetical mechanism takes a great deal of effort, as well as Gershenfeld's mention of implicitness. Otto & Day also explain that this comes from a deficit in modelling in students' background; they suggest starting simply as a means of dealing with this initial difficulty, and that mastery comes with practice.

The third modelling textbook, written by the CSMG primarily for those in healthcare [6], does venture a definition of modelling:

> "In this book the word "model" means a simplified representation of a real-world situation used to help answer a specific question." ( [6], p.4).

Note that the authors clarify that this definition pertains to this book only, implying that there may be other definitions of modelling. The authors in this case are mathematicians who have come to interdisciplinary work later in their careers, and are cognizant of the fact that common words have disparate meanings in different communities. This definition includes the need of a model to answer a specific question (which was alluded to in the previous definitions) without the emphasis on the question needing to be specific. The authors go on to explain what makes a good model, re-iterating Gershenfeld's idea that there is no "best" model:

> "[A] good model has as low a complexity as possible while retaining the details necessary to approach the specific question the model is designed to examine. In general, models with a focused question and a limited number of conditions are more likely to be useful [...] there is no such thing as a unique "best" model for a given problem. In fact, in most cases, more than one model discussed in this book is applicable in solving a single question. In these cases different modelling methods are often complementary, with the best results obtained through an approach that integrates multiple methods. In general, modelling is most convincing when various different kinds of models lead to the same conclusion." ( [6], pp. 4-5).

In contrast to Howison and Fowler who value the predictive power of the model, the CSMG speak of a comparison of methods to lend validity to the model. This book also provides a flowchart of the modelling process.

| Box 2.1: | Seven Steps to Modeling a Biological Problem |
|---|---|
| *Step 1*: | *Formulate the question*<br>What do you want to know?<br>Describe the model in the form of a question.<br>Boil the question down!<br>Start with the simplest, biologically reasonable description of the prob. |
| *Step 2*: | *Determine the basic ingredients*<br>Define the variables in the model.<br>Describe any constraints on the variables.<br>Describe any interactions between variables.<br>Decide whether you will treat time as discrete or continuous.<br>Choose a time scale.<br>Define the parameters of the model.<br>Describe any constraints on the parameters. |
| *Step 3*: | *Qualitatively describe the biological system*<br>Draw a life-cycle diagram for discrete-time models involving multiple events per time unit.<br>Draw a flow diagram to describe changes to the variables over time.<br>For models with many possible events, construct a table listing the outcome of every event. |
| *Step 4*: | *Quantitatively describe the biological system*<br>Using the diagrams and tables as a guide, write down the equations.<br>Perform checks. Are the constraints on the variables still met as time passes? Make sure the units of theRHS equal those on the LHS.<br>Think about whether results from the model can address the question. |
| *Step 5*: | *Analyze equations*<br>Start by using the equations to simulate and graph the changes to the system over time.<br>Choose and perform appropriate analyses.<br>Make sure that the analyses can address the problem. |
| *Step 6*: | *Checks and balances*<br>Check the results against data or any known special cases.<br>Determine how general the results are.<br>Consider alternatives to the simplest model.<br>Extend or simplify the model, as appropriate, and repeat steps 2-5. |
| *Step7*: | *Relate the results back to the question*<br>Do the results answer the biological question?<br>Are the results counterintuitive? Why?<br>Interpret the results verbally, and describe conceptually any new insights into the biological process.<br>Describe potential experiments. |

Table 2.1: Otto & Day's Box 2.1 - A seven step mathematical modelling cycle.

### 2.1.2 Issues with Modelling: Teaching Approach

Along with the variation in the definition of modelling, there is variation in the approach to teaching it. The first category of authors, aiming at the students of mathematics, approach modelling by making use of case studies as a means of exposition. Illner and Howison introduce case studies almost immediately. There is some overview of the modelling process discussed in Fowler's text, before case studies are presented. Briggs' goes further with a discussion on problem-solving heuristics first, followed by case studies that highlight different heuristics or steps in the modelling cycle. In contrast to the case studies approach, the authors aiming primarily at those outside of the mathematics field address the teaching of modelling by focusing on techniques or modelling by design. Gershenfeld discusses the techniques used for solving models including analytic, numeric and observational modelling techniques, thus only addressing one area in the modelling cycle (solving the model). After a description of the overall modelling process Otto & Day also discuss several modelling techniques for solving problems arising in Biology. The exception to this is the CSMG who, like Briggs, make use of case studies while highlighting the different aspects of the modelling process when looking at each case.

### 2.1.3 Issues with Modelling: from Novice to Expert

The textbook literature reveals a dichotomy of approach to modelling: case studies or modelling by design. There are also varying definitions of modelling, and in some cases no definition at all. This lack of precision when defining modelling is understandable but begs the question, are all modellers describing and focusing on the same thing when they use the term mathematical modelling? While in many cases these authors have addressed some of the difficulties of the student, what they have not addressed are the skills necessary to move from novice to expert. It is therefore necessary to continue on to the Ictma modelling articles to see how the spectrum from novice to expert is addressed.

Crouch and Haines outline several cognitive and meta-cognitive differences between experts and non-experts [12,18,19]: three of them will be highlighted here. The first difference between experts and novices is their approach to problems. Experts begin with analysis and a plan. They constantly return to re-examine the problem and re-define variables. Novices on the other hand tend to plunge in, go straight to equations and stick to their original thoughts regardless of where these thoughts lead [20–22]. Experts tend to use forward thinking more often and better than their novice counterparts, who are more likely to work backwards from the solution [22–24]. This has serious implications for modelling, as using the results to create the model results in a biased model. The third and final difference is a meta-cognitive one. Experts not only have better domain-specific knowledge, but this knowledge is also better inter-connected. This superior knowledge causes experts to focus on underlying principles. Novices on the other hand have knowledge that is loosely connected and tend to focus on the surface features of the problem rather than the underlying principles [25, 26]. It must be noted that these skills of expertise described here take a relatively long time to acquire [27].

This look at the textbooks and the literature on modelling education provides a general idea of the modelling process and the differences we expect between the novice and the expert modeller on a cognitive and meta-cognitive level. However, looking at them together still does not provide a novice modeller with enough information to move along the path to expertise. There are various definitions of modelling available making it difficult to ensure that experts and novices are thinking of the same process when talking about mathematical modelling. The cognitive deficits of the novice modeller are cited, but as these aspects of expertise take time to develop, the novices cannot force themselves to be more expert at organising their thoughts, for example. There is also no discussion evidenced of how to get unstuck, except via simplification of the model. This creates problems in the case when it is uncertain how to simplify the model, and also if the model already seems to be in its simplest form.

To address the issue of being stuck I turned to the literature available on problem-solving. Problem-solving is not identical to mathematical modelling. As will be seen in the discussion of the literature in Section 2.2, there is a range of opinion about problem-solving problems being problematic *i.e.* it is only a problem if it causes the solver to be stuck [28, 29]. Our above look at the varying modelling definitions and ideas has not revealed that modelling problems have to be problematic in nature. However, since modelling problems do tend to be messy [7], and the formulation step is important and not always obvious [15, 16], then modelling problems can have problematic aspects to them, causing the modeller to be stuck in the modelling process.

The issue of moving from a state of being stuck to becoming unstuck has been one of the main focuses of problem-solving literature in the last few decades. In particular, problem-solving literature has pushed past initial descriptions of problem-solving by design, which focuses on the cognitive and logical processes only, to identify the extra-logical processes involved in mathematical modelling. Therefore, I looked at how problem-solving was addressed in the earlier literature, and how it is approached now, focusing on the issue of moving forward from a state of being stuck.

## 2.2 Problem Solving Literature

There are two main approaches to problem-solving. The first approach focuses on the cognitive aspects of problem-solving, which are the "mental operations that support people's acquisition and use of knowledge" [30, 31], in other words, the way we think. The word "acquisition" suggests that deliberate effort is used in achieving the knowledge. This first approach is similar to what we have seen in the literature on mathematical modelling in Section 2.1.3. The second approach to problem-solving pushes past the logical (the use of deliberate reasoning), to the extra-logical (the use of subconscious processes) in order to deal with the issue of being stuck. The term extra-logical encompasses those mental processes that are not achieved through deliberate conscious thought, which includes creativity, intuition, and illumination also known as the AHA! experience [32, 33].

### 2.2.1 Problem Solving: by Design

Looking at Polya's *How to Solve It* [34], Polya gives a four-stage description of the problem-solving process that is similar in many aspects to the mathematical modelling process (see Table 2.2).

Many mathematicians will agree this is indeed what they do when solving problems. However, Schoenfeld explains in his book *Mathematical Problem Solving* [35] that while these steps are a description of the problem-solving process, they are not enough to use as a prescription for how to do mathematical problem-solving. This is due to the fact that there are not enough details nor instructions for the novice to be able to solve any given mathematical problem:

> "1. Typical descriptions of heuristic strategies, for example, "examining special cases," are really labels for categories of closely related strategies. Many heuristic labels subsume half a dozen detailed strategies or more. Each of these more precisely defined strategies needs to be fully explicated before it can be used

| | |
|---|---|
| First.<br>You have to *understand* the problem. | UNDERSTANDING THE PROBLEM<br>What is the unknown? What are the data? What is the condition? Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory? Draw a figure. Introduce suitable notation. Separate the various parts of the condition. Can you write them down? |
| Second.<br>Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a *plan* of the solution. | DEVISING A PLAN<br>Have you seen it before? Or have you seen the same problem in a slightly different form?<br>Do you know a related problem? Do you know a theorem that could be useful?<br>Look at the unknown! And try to think of a familiar problem having the same or a similar unknown.<br>Here is a problem related to yours and solved before. Could you use it? Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?<br>Could you restate the problem? Could you restate it still differently? Go back to definitions.<br>If you cannot solve the proposed problem try to solve first some related problem. Could you imagine a more accessible problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or the data, or both if necessary, so that the new unknown and the new data are nearer each other? Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem? |
| Third.<br>*Carry out* your plan. | CARRYING OUT THE PLAN<br>Carrying out your plan of the solution, check each step. Can you see clearly that the step is correct? Can you prove that it is correct? |
| Fourth.<br>*Examine* the solution obtained. | LOOKING BACK<br>Can you check the result? Can you check the argument? Can you derive the result differently? Can you see it at a glance? Can you use the result, or the method, for some other problem? |

Table 2.2: Polya's description of the different stages of problem-solving.

> reliably by students.
>
> 2. The implementation of heuristic strategies is far more complex than at first appears. Carrying out a strategy such as "exploiting an easier, related problem," for example, involves six or seven separate major phases, each of which is a potential cause of difficulty. Training in the use of the strategy must involve training in all those phases, and the training must be given with at least as much care and attention as is given to standard subject matter. In general attempts to teach such strategies have not been adequately precise or rigorous.
>
> 3. Although heuristic strategies can serve as guides to relatively unfamiliar domains, they do not replace subject matter knowledge or compensate easily for its absence. Often the successful implementation of a heuristic strategy depends heavily on a firm foundation of domain-specific resources. It is unrealistic to expect too much of these strategies." ( [35], p.73)

This again is reminiscent of the issues already addressed in the discussion of the mathematical modelling literature. Schoenfeld's research is focused on doing, understanding and teaching mathematical problem-solving. He shows how for a particular problem we would have to break down the given heuristic in order to teach it. The intent of his work is to develop a prescription as opposed to a description (such as Polya's work), for mathematical problem-solving.

Schoenfeld addresses the differences between novice and expert problem-solving: even though novice problem-solvers have the necessary resources, often they do not know to access them for a given problem. Schoenfeld explains that this is an issue of control: knowing what resources to access and when, and that this is integral to being able to be a successful problem-solver. This control, or ability to understand your own thought processes is referred to as metacognition, first defined by Flavell as "one's knowledge concerning one's own cognitive processes or anything related to them." [35, 36]. Schoenfeld also addresses the fact that expert problem-solvers understand the underlying processes of the problem while novices do not, as was discussed in the literature on the difference between novice and expert modellers. Schoenfeld concludes that:

> "A huge amount of work lies ahead. We have barely begun to scratch the surface of any of the categories described in Part 1: resources, heuristics, control and

belief. Even in the best laboratory situations, these are complex and elusive."
( [35], p.375).

So how do the researchers of modelling go on to grasp the elusive aspects of modelling? The first step is to recognise that mathematical problem-solving is qualitatively similar to mathematical creativity. The framework for mathematical invention has been referred to throughout the literature on mathematical problem-solving. More recently in his work on the AHA! experience, Liljedahl [33] notes that '[m]athematical problem-solving is synonymous with invention and discovery' ( [33], p.15).

### 2.2.2 Problem Solving: Incorporating the Extra-Logical

In his work *The Mathematician's Mind: The Psychology of Invention in the Mathematical Field* [32], Hadamard looks at how mathematicians create mathematics. He uses the works of Poincaré to explain that there are four stages to mathematical invention:

1. Preparation: in this stage the mathematician is using deliberate conscious effort to come to a solution.
2. Incubation: where the mathematician, if stuck in his/her deliberate efforts, stops working on the problem
3. Illumination: where the unconscious mind solves the problem and the answer becomes available to the conscious mind (also referred to as the AHA! experience)
4. Verification: the mathematician then verifies that the solution is correct by deliberate means.

In later problem-solving literature, there is an extension of Polya and Schoenfeld's frameworks to include the creative aspects of problem-solving informed by Hadamard's work. The creative aspect applied particularly to when the problem-solver was stuck (see Hadamard's step 2 above). This amalgamation of the two frameworks of Polya and Hadamard are evidenced in Mason *et al.*'s work *Thinking Mathematically* [28].

Mason's framework of modelling begins with the premise that everyone can start by specialising. This is similar to the initial simplification recommendation in the modelling literature. Generalising or drawing inferences from the specialised cases follows this step. Mason also recommends making notes of thoughts feelings and ideas, which ties into metacognition, the idea of monitoring your own thoughts. In particular, if you are stuck he recommends acknowledging this and suggests ways to move from the state of being stuck to unstuck.

> "STUCK!
> Whenever you realize that you are stuck, write down STUCK! This will help you to proceed, by encouraging you to write down why you are stuck. For example:
> I do not understand...
> I do not know what to do about...
> I cannot see how to...
> I cannot see why...
>
> AHA!
> Whenever an idea comes to you or you think you see something, write it down. That way you will know later what the idea was. Very often people have a good idea, but lose it subsequently and cannot recall it. In any case it feels good to write down AHA! Follow it with:
> Try...
> Maybe...
> But why...
>
> CHECK
> Check any calculations or reasoning immediately
> Check any insight on some examples (specializing)
> Check that your resolution does in fact resolve the original question.
>
> REFLECT
> When you have done all that you can or wish to, take time to reflect on what happened. Even if you do not feel that you got very far, it helps to write up what you have done so that you can return to it freshly and efficiently at some later date. It is also the case that the act of summarizing often releases the blockage. There are several things worth noting particularly:
> Write down key ideas
> Write down key moments that stand out in your memory
> Consider positively what you can learn from this experience." ( [28], p.19)

Mason considers the reflection step "the most important activity to carry out" ( [28], p.131) as it helps to develop your metacognition or internal monitor. These last two steps of

Mason's framework are similar to the final step in Polya's model. Mason additionally acknowledges that the problem-solver will possibly get stuck and require some extra-logical illumination or "AHA!" moment as seen in Hadamard's model. He also incorporates feelings in the final step that is not evidenced in Polya's work.

Perkins' work *Archimedes Bathtub: The Art and Logic of Problem Solving* also deals with the problem-solving issue of being stuck and how to deal with it. In fact, Perkins explicitly addresses only problems where you are stuck initially. He calls these problems "unreasonable" and they are the problems that require breakthrough thinking or illumination to solve, that is, you cannot approach them by deliberate logical effort:

> 'The surface pattern of breakthroughs reflects not underlying mental processes but the underlying structure of the problems themselves. To put it roughly, many problems are reasonable: They can be reasoned out step by step to home in on the solutions. But certain problems are unreasonable: They do not lend themselves to step-by-step thinking. One has to sneak up on them.' ( [29], p.22)

Perkins deals with four different ways that one may feel stuck and the solution for becoming unstuck. The pertinent solution depends on why you are stuck and includes: brainstorming, detecting hidden clues by looking in a different place or more carefully for incongruous features, reframing the situation, backing up to an earlier point and taking a different path from there by bracketing off the current approach and trying something else [29].

This range of literature on problem-solving began with a framework very similar to that currently seen in mathematical modelling, dealing with the cognitive and meta-cognitive aspects of problem-solving. This can be described as problem-solving by design. Later work by Mason and Perkins has moved on to look at aspects outside of the cognitive, to those subconscious processes that help the problem-solver to become unstuck. If mathematical modelling is synonymous with problem-solving, and hence creativity, then these extra-logical processes are equally transferable to the modelling issue of being stuck. Even if problem-solving and mathematical modelling are significantly different, do expert modellers employ these skills for dealing with being stuck? Or are the experts never stuck and travel along the modelling cycle with relative ease? Finally, are the extra-logical processes of getting unstuck the only non-cognitive aspects that hinder novices from progressing?

## 2.3 Psychology of Modelling

From the look at the modelling and problem-solving literature it is clear that mere cognitive and logical processes are not enough to make one a successful modeller. If this were the case then simply studying more mathematics and looking at a flow chart of the cognitive processes required for modelling would be enough to gain expertise. This has not proven to be the case as the modelling literature indicates that modelling is difficult to define, difficult to teach, and can be overwhelming. This led me to an investigation into possible psychological issues that could affect the novice modeller.

To delve into some of the psychological issues that may be associated with mathematical modelling. I examined the works of Tobias, Pink and Csikszentmihalyi who discuss the themes of anxiety, motivation and flow. These themes are found to interconnect and provide a backdrop on some of the non-cognitive issues that may arise when modelling.

### 2.3.1 Math Anxiety

Mathematical modelling has no closed form solution, no set solution framework and requires tools from many areas of mathematics. These characteristics may lead modelling to trigger math anxiety, especially among the novice modellers. Tobias in her work *Overcoming Math Anxiety* claims that "all people endure some mathematics anxiety, but it disables the less powerful [...] more"( [13], p.9).

Tobias explains that math anxiety is the feeling that one can not do a problem and would "*never* go any further in mathematics"( [13], p.50). This first feeling of failure is "instant and frightening" and feels like a "sudden death"( [13], p.50). Math anxiety is usually caused initially by some significant, traumatic event in the math-anxious person's mathematics life. Tobias [13] tells us that "[m]ath-anxious adults can recall with appalling accuracy the exact wording of a trick question or the day they had to stand at the blackboard alone, even if these events took place thirty years before"( [13], p.38). She explains that math anxiety creates feelings of paranoia, fear of asking for help, shame, guilt and feelings of fraud. Math anxiety stems from several issues including the belief that errors are shameful and the attitude that mathematics ability is inborn. In contrast, positive psychology, focused on "well-being, contentment, and satisfaction (in the past); hope and optimism (for the future); and flow and happiness (in the present)" [37] is associated with mastery. Thus to identify if math anxiety is an issue in mathematical modelling, there is a need to investigate the feelings of the expert, intermediate and novice modeller.

There are several issues highlighted in Tobias' work that create math anxiety. The first cause is ambiguity in mathematics. Tobias notes, "Though mathematics is supposed to have a precise language [...] mathematical terms are never wholly free of the connotations we bring to words and these layers of meanings may get in the way. The problem is not that there is something wrong with math; it is that we are not properly initiated into its vocabulary and rules of grammar" ( [13], p.54). Modelling problems have been described as being messy and coming from the real world [4, 7, 15, 16], which creates a new difficulty with the language of the field of the problem as well as those language issues raised by Tobias. Combined with this is the fact that the definition of mathematical modelling itself is ambiguous, thus we suspect that this aspect of modelling may create anxiety, especially for students that have been successful with other fields of mathematics.

The second cause of math anxiety is dubbed the "dropped stitch," which is the belief that "if we haven't learned something so far, it is probably because we can't" ( [13], p.62). This ties in with the belief that understanding and ability is fixed [13, 38]. The belief that ability is fixed and the feeling of not being able to do a problem or learn a skill suggests a low perceived self-efficacy. Bandura defines perceived self-efficacy as "people's beliefs about their capabilities to produce designated levels of performance" [39]. He explains that since people with low self-efficacy "view insufficient performance as deficient aptitude it does not require much failure for them to lose faith in their capabilities" [39]. Weiner discusses this same issue, "attributions of failure to lack of ability [...] produce low expectancies of future success (tied to the stability dimension of causality), low self-esteem (linked with the locus dimension), and humiliation and shame (because these are perceived as uncontrollable)" [40]. This lack of belief in self and belief in one's ability to learn has serious ramifications in terms of modelling since modelling covers a broad range of topics and subject areas. Lack of proficiency in an area, where one believes they can no longer learn it, makes modelling extremely difficult if not impossible.

A third cause of math anxiety pertaining to modelling is a distrust of intuition. Tobias explains "math-anxious people seem to have little or no faith in their own intuition" ( [13], p.62). This is a factor if intuition plays a large role in mathematical modelling, as hinted by Otto & Day. In this analysis, it is necessary to identify if this is the case.

Tobias also notes that students who expect to find "the right answer" can feel anxious when that right answer does not present itself. Nevertheless she states, "This emphasis on right answers has many psychological benefits. It provides a way to do our own evaluation on the spot and to be judged fairly whether or not the teacher likes us" ( [13], p.67). For the successful mathematics student, finding the exact answer may be a reassurance rather than a cause of anxiety (which is why they are successful mathematics students). These students may have the problem of "the right formula [becoming] a substitute for thinking" ( [13], p.69) They may therefore feel anxiety when the problem does not yield a clear closed-form solution, as is the case with mathematical modelling.

### 2.3.2 Motivation and the Type I Personality

Daniel Pink discusses motivation in his book *Drive: The Surprising Truth About What Motivates Us* [41]. He establishes two distinct types of motivation and the people who tend to each type. The Type X person is extrinsically motivated, concerning themselves "less with the inherent satisfaction of an activity and more with the external rewards to which that activity leads"( [41], p.77). In contrast, the Type I person is intrinsically motivated and primarily concerns themselves with inherent satisfaction primarily. As well as having intrinsic and extrinsic motivation, one can attribute success to internal or external factors, which is what Weiner [42] calls the locus of control, whereby success is either controlled internally (*e.g* skills, ability, dedication) or by external aspects (*e.g.* luck, circumstances, difficulty of the task) [42].

Pink explains that the most successful people are "working hard and persisting through difficulties because of their internal desire to control their lives, learn about the world and accomplish something that endures" ( [41], p.79). This is connected to self-efficacy and Bandura's work, as he explains that "[p]eople with high assurance in their capabilities approach difficult tasks as challenges to be mastered rather than as threats to be avoided. Such an efficacious outlook fosters intrinsic interest and deep engrossment in activities" [39]. In other words, the Type I personality is what we expect of experts in modelling. In examining the differences between the expert and the novice we must look at motivation as this plays a role in expertise. Pink further elaborates on three main elements of Type I behaviour. We will look at all three of these elements in relation to our investigation of the difference between the novice and the expert.

The first element of intrinsically motivated behaviour is autonomy. Pink quotes designer Sagmeister as saying, "Autonomy over what we do is most important. The biggest difference between working for other studios and running my own has been the fact that I can choose what job we take on and what product, service or institution we promote" ( [41], p.104). How does autonomy affect the modeller at all levels of expertise? Do expert mathematical modellers have autonomy? Can the same be said for their non-expert counterparts, and if not does this affect their level of expertise? Note here that autonomy is not synonymous with isolation. Autonomy "is not the rugged go-it-alone, rely-on-nobody individualism of the American cowboy. It means acting with choice which means we can be both autonomous and happily interdependent with others."( [43]; [41], p.90). As autonomy is an element of Type I behaviour, then we will identify whether experts exhibit this mix of autonomy and interdependence.

Mastery is the second element of Type I behaviour. It is defined in this work as "the desire to get better and better at something that matters" ( [41], p.111). Dweck describes different types of motivation as adaptive or maladaptive, explaining that "[t]he adaptive ("mastery-oriented") pattern is characterized by challenge seeking and high, effective persistence in the face of obstacles [...] In contrast, the maladaptive ("helpless") pattern is characterized by challenge avoidance and low persistence in the face of difficulty" [44]. Mastery is linked closely with the concept of flow where "the relationship between what a person had to do and what he could do was perfect. The challenge wasn't too easy nor was it too difficult. It was a notch or two beyond his current abilities, which stretched the body and mind in a way that made the effort itself the most delicious reward" ( [41], p.115,). This definition of flow comes from Csikszentmihalyi [45] who links flow to creativity. Pink goes on to state that several companies have "realized that creating flow-friendly environments that help people move toward mastery can increase productivity and satisfaction at work"( [41], p.117). Our study attempts to describe the landscape from the novice modeller to the expert. It is by definition an investigation of different levels of mastery.

Pink explains that mastery is a mindset, and is rarely achieved if you are extrinsically motivated; is a pain and requires perseverance and determination; and is an asymptote, and thus for the most part unattainable. Our study will use these features of mastery to identify whether the experts, and non-experts exhibit these characteristics of mastery. We also want to identify if flow emerges as a theme among modellers.

The final element of the Type I behaviour is that of purpose. Pink claims "the most deeply motivated people – not to mention those who are most productive and satisfied – hitch their desires to a cause larger than themselves" ( [41], p.133). This statement prompts us to consider if there is a higher purpose among the expert modellers.

### 2.3.3 Creativity and Flow

Our look at work from Tobias on anxiety and Pink on motivation address issues that we expect to affect the novice modeller or help describe the landscape from novice to expert. Finally we look at Csikszentmihalyi's work on creativity, a trait that should primarily reside with the experts if anywhere. In his work *Creativity: Flow and the Psychology of Discovery and Invention* [45] Csikszentmihalyi defines creativity as "any idea, act or product that changes an existing domain, or that transforms an existing domain into a new one" ( [45] p.28). Here it is important that creativity has an impact on the cultural matrix. He speaks of creativity being a marriage of the individual's novel work, the acceptance of the field of experts and the impact on the domain of the creative work. He distinguishes this definition of creativity by calling it creative with a capital C. The first question we must address on this front is whether mathematical modellers are creative in the Csikszentmihalyi sense.

What aspects and personality traits affect creativity? These are numerous and would require a complete discussion of Csikszentmihalyi's work. Instead I highlight a few aspects that I believe may be pertinent to the mathematical modeller. The first of these is curiosity. Csikszentmihalyi establishes that, "without a good dose of curiosity, wonder and interest in what things are like and in how they work, it is difficult to recognize an interesting problem" ( [45], p.53). With this in mind I will be looking for evidence of curiosity among participants of all levels of expertise.

The second aspect of interest to this work is the trait of adaptability. Modelling is varied and broad and may require the ability to deal with many situations. Csikszentmihalyi cites adaptability as being one characteristic of the creative personality: "Creative individuals are remarkable for their ability to adapt to almost any situation and to make do with whatever is at hand to reach their goals" ( [45], p.51). Do the successful mathematical modellers exhibit these traits? Do they adapt well? Do they recognize the resources around them? Do they make intelligent use of all these resources?

The final aspect that I looked at was that of flow. This was mentioned earlier while looking at Pink's work on motivation. Csikszentmihalyi explains that creative personalities have a multiplicity of traits that are often contradicting, however he states "in one respect they are unanimous: They all love what they do." ( [45], p.107). He goes on to explain that this love of what they do comes from the experience of flow in their respective domains when working on a problem. Flow has several essential components to it: clear goals, immediate feedback, a balance between challenges and skills, a situation where action and awareness are merged, lack of distraction, no worry of failure, no consciousness of self, no awareness of time and autotelic activity ("the activity is its own reward" [41]).

Pink explains in a nutshell why flow is important for any group: "When what they must do exceeds their capabilities, the result is anxiety. When what they must do falls short of their capabilities, the result is boredom." ( [41], p.119). This brings together all three works examined for the psychology of mathematical modelling.

Mathematical modelling deals with open-ended problems and is a broad and varied area. It has the ability to be overwhelming as well as intriguing. This suggests that the skills required are more varied than the simple ability to do mathematics well. What other aspects go in to being a good modeller? What motivates the mathematical modeller? How does the modeller feel when working through difficult problems? Does he/she experience anxiety when approaching a modelling problem? And if so how is this dealt with?

## 2.4 Summary

This brief review of modelling literature identifies that there are still several gaps in the modelling literature. For instance, there is no agreed upon definition of mathematical modelling in the literature. Additionally, there is a dichotomy in the approaches for teaching modelling and it is not clear what the best way to do this should be. Finally, while the cognitive and metacognitive issues are well-researched, there is less emphasis on those non-cognitive areas (subconscious, emotional and psychological processes) that are associated with modelling.

A look at the problem solving literature highlights a shift in focus from a look at cognitive processes to extra-logical or subconscious processes. In particular, illumination or AHA! experiences are identified as integral when dealing with being stuck. Although mathematical modelling is not identical to problem solving there is some overlap, and this shift in focus in the problem solving literature suggests an investigation into the necessary processes for dealing with being stuck when doing mathematical modelling.

The differences along the spectrum from novice to expert are not well-researched at the non-cognitive level. However, an examination of some of the cognitive, developmental, and motivational literature suggests that non-cognitive issues (particularly emotional and psychological) have a role to play in developing cognitive ability. Anxiety, self-efficacy, motivation and flow all are related to success and mastery. This combined with the above-mentioned issues raises several questions of interest. In the following chapter, I will discuss some of those pertinent questions and describe how I went about trying to seek answers to them.

# Chapter 3

# Methodology

Having described the main literature on mathematical modelling to give a glimpse of how modelling is viewed in the applied mathematics community, it is clear that there are several questions outstanding.

## 3.1 Research Questions

**Research Question 1**

The mathematical modelling textbooks have highlighted that definitions of mathematical modelling are varied and in some cases non-existent [4–7, 15–17]. Approaches to exposition on the subject are dichotomous, split between case studies and focusing on teaching solution techniques. This leads is to our first research question:

> *What IS mathematical modelling? What do the experts mean when they talk about modelling, and is it the same as what the non-experts mean?*

This has implications for doing and teaching modelling. If we have a working definition of what we're talking about when using the term modelling, then we can better understand and teach it. It also helps us to communicate with others who use the term if we know the different definitions of modelling they could be applying.

**Research Question 2**

An investigation of the problem solving literature highlights extra-logical mechanisms for getting unstuck. Modelling experts in the field may use these subconscious processes as well, though they are not evidenced in the modelling literature. This leads us to the second research question:

> *How do expert modellers move from a state of being stuck to becoming unstuck? Are extra-logical processes used, or do expert modellers have other strategies for doing so?*

The answer to this question will aid in doing mathematical modelling. The novice modeller may be able to employ the techniques used by the experts when they are stuck in one stage of the modelling process.

#### Research Question 3

Crouch and Haines' work [18] indicate cognitive and meta-cognitive differences between novices and experts but do nothing to address the non-cognitive issues, particularly how to get unstuck and what psychological areas affect modelling. A look at the math anxiety literature indicates a need to probe into the feelings, attitudes and beliefs of the participants of the modelling study. In this way we can identify how math anxiety hinders the ability to do modelling well. The literature on motivation highlights that motivational factors are directly correlated to expertise. In the study we hope to see evidence of intrinsic versus extrinsic motivation being an indicator of expertise. Qualities of autonomy, mastery and purpose will aid in identifying intrinsic motivation in the participants. Finally a look at creativity suggests that these psychological factors may be contributing factors to the differences between novices and experts. This leads us to our final research question:

> *What are the differences between the novice and the expert modeller? Particularly what are the non-cognitive differences?*

This final question has implications for the teaching of mathematical modelling. Many of the known differences between the expert and the novice modeller cannot be taught and need time to be developed. Nonetheless it is possible that some differences revealed could be used to help the novice move along the spectrum towards becoming an expert.

## 3.2 Overview of the Study

In response to these questions, I tailored this study in order to target different groups of expertise, and answer the three research questions that have emerged from this look at the literature. In order to answer the questions addressed above, I conducted a qualitative study with 78 people who do modelling at some level. A qualitative as opposed to quantitative approach was used as I was trying to establish a fundamental understanding of mathematical modelling and the people who partake in it. The point of this work is not to create a formula for doing mathematical modelling, or to establish that 400 people believe modelling is one particular thing. The aim of the project is to understand different variations in modelling and to understand the nuances in the modelling world. This requires a qualitative approach.

The literature on problem solving informed my tools of data gathering. In his study to understand novice problem solvers Schoenfeld [35] does a clinical study with his participants. However Liljedahl's [33] work informs us that a clinical trial does not always help us to identify the internal aspects of the problem solving process. I felt that the internal aspects were worthy of study, which meant that a clinical trial would not make sense here. Hadamard [32] and Liljedahl both make use of questionnaires, while Schoenfeld and Liljedahl both use observations to inform their studies. Therefore, observations, interviews and questionnaires were my tools of data gathering.

In an effort to answer the research question regarding the differences between the expert and the novice modeller, I needed to understand the modelling process from the point of view of the expert, the novice and those who are in-between. How do we identify the different groups and find out what modelling entails for them? In order to do this I needed to determine who would qualify as an expert or a novice in the field of modelling. This led me to look at the Dreyfus model of expertise [46, 47] for a description of the mental skills expected as expertise increases.

In the 1970's the Dreyfus brothers [46] looked at highly skilled experts from pilots to chess grand masters and did seminal research on how people attain and master skills. They found that one fundamentally changes how one perceives the world beyond just gaining new skills or knowledge. They developed a theory which has been applied to diverse fields including nursing, education, computer programming, sport and others. In particular, they developed five categories defined by universal characteristics:

> Novice: has no skill; needs rules; has no self-confidence; not interested in learning just completing; progresses by relying only on rules.
>
> Advanced Beginner: can start tasks on their own; little self-reflection; can be mentored; has no idea of big picture or overall principles; can learn via guided

experience.

Competent: can develop novel conceptual models; troubleshoots: “Is this right?”; work based on experience and planning; is often overwhelmed by the choices available; can learn by doing.

Practitioner: Needs the big picture; self-reflects: “Can I do it better the next time?”; has sufficient experience to learn from discussion and case studies; can learn by seeing and listening.

Expert: Intuits solutions; may have difficulty explaining what he/she does as it has become automatic.

These characteristics helped to identify which participants will be targeted to fill the role of expert and novice and those between. While the expert category and novice category are obvious from the outset (one group has had significant experience and the other has had none), traversing the other three groups is a little more subtle and we expect to see some overlap in abilities from one level to the other. To place a person in any of the given middle categories may require a prior assessment of their work, which is not feasible.

I therefore split the participants of the study into four major groups: expert, intermediate, novice and complete novice, each of which represents a different level of expertise. In this chapter I will discuss each group explaining who they are, how they were accessed, what data was acquired and how. I will then give an overview of how the data was analysed, before going on to discuss the results of this study in the following chapters.

## 3.3 Group 1: Expert Modellers

The first group that I chose to work with were experts in the field. This was informed by Hadamard's and Liljedahl's work [32, 33]. From the Dreyfus model discussed above, we know that experts intuit solutions and much of what they do has become automatic [46]. Therefore the obvious choice of experts was professors of modelling. These professors must be active mathematical modellers, preferably being well recognised in the field, as this is an indicator of their expertise. If they also taught mathematical modelling this would be an added bonus, as they would have insight into their novice counterparts. The experts that participated came from prestigious universities in Canada, the United States and the United Kingdom, including UCLA, Oxford and Duke.

Several of these experts mention and regularly take part in annual modelling study groups hosted by many different universities and societies. Preceding the study group there is usually a camp for the graduate students where a mentor leads the students through modelling problems. This is followed by a week-long study group in which faculty and students work together. The format for these events is as follows: before the event organizers work with members of industry to find suitable problems, on the first day the industrialists present their problems, participants then choose their favourite problem and work in teams, often parcelling out the work to have it run more efficiently. While everyone in the group is entitled to opinions and encouraged to voice them, all opinions must be defended mathematically. At the end of the week some analytical or numerical mathematical models and solutions are presented to everyone attending, in answer to the given industrial problem.

I contacted my first experts by attending the Graduate Student Mathematical Modelling Camp (GSMMC), hosted by the Rensselaer Polytechnic Institute. I worked on one of the modelling problems and I was able to see first hand how the process worked. Here professors mentored graduate students in the art of modelling over the course of a week, offering support (or not) as needed throughout the camp. These mentors were my first targets for the study. I acquired interviews from the four mentors that assisted the students in the GSMMC.

I then emailed a list of 13 experts I received from my supervisor who also works in the field of modelling. Of the 13 people emailed, 7 agreed to work with me on the study. These 7, along with the initial 4 from the GSMMC brought my total experts up to 11. Along with these I was also able to interview 3 experts who visited SFU to give presentations on their areas of modelling expertise. This brought my total up to 14. At this point I noticed no new data emerging so I stopped interviewing experts.

In the case of the experts the data consisted of the responses to ten interview questions. Interview questions seemed appropriate as they gave some flexibility in being able to ask follow-up questions. This is pertinent as Dreyfus & Dreyfus warn us that experts may have difficulty explaining what they do as it has become automatic [46]. The interview questions were informed by Hadamard's survey [32] but were adapted to address my own research questions. The interview addresses different aspects of the modelling process, and questions were ordered to mimic the order of the steps in the modelling process. I therefore started with questions on definitions and first thoughts upon seeing a problem and worked through different areas of modelling up to and including checking the model. For this reason the questions were always asked in the same order, but follow-up questions were included if necessary to glean further information from the responses. Question eight was adapted slightly as indicated by feedback from the initial interviewees. The responses to these questionnaires will be discussed and analysed in Chapter 4.

## 3.4 Group 2: Intermediate Modellers

The Dreyfus model describes three distinct levels between the expert and the novice [46], ranging from advanced beginner to competent. While the characteristics of the expert and novice were extreme and therefore easy to identify, characteristics of the intermediate levels were more subtle and not always easily identifiable from the outset. I therefore diverged from the Dreyfus model and dubbed all participants intermediates if they were neither expert nor novice. The intermediate participants were made up of 11 of SFU's applied mathematics graduate students and post-doctoral fellows. These participants had varied backgrounds coming to SFU from six different countries including, Canada, the United States, and China. They were also varied in their applied mathematics interests.

I requested interviews with the intermediates in person, as I had direct access to them. Many of these students I had worked with in a voluntary, weekly problem-solving session held by and for the applied mathematics graduates. Oftentimes after an intermediate was interviewed they would suggest another possible participant for the study whose work they were more familiar with than I. While several of my colleagues were willing to help a few declined, explaining that they had not done any mathematical modelling. (It is unlikely that they had no modelling experience at all, which makes this an interesting response. I assume that their definition of modelling was not the same as mine, since we have seen that modelling has varying definitions.) As several of these intermediates took part in the weekly problem-solving session they had this as a common background.

This problem solving session occurred once a week, usually on Friday evenings, for the applied mathematics graduate students at SFU. It was voluntary, but students were expected to participate if possible. A mentor provided a problem (and sometimes the students did as well), which was oftentimes a modelling problem that the group of graduate students then attempted to solve. There was no set time limit, with some problems spanning several weeks. While the session was mentored, it was not as structured as the modelling camps described in the previous section. Students were expected to self-organise and decide what they worked on. This often led to several groups working at different paces, on the same aspects of the problem. This common experience gave me something to fall back on if the intermediates interviewed needed examples to help prompt their responses.

Once again the data consists of the responses to the interview questions. Initially I tried to include graduate students that participated in the GSMMC (see above Section 3.3) with the use of questionnaires to gather data. This resulted in varying qualities of response, due to the range of expertise in these graduate students. I realised that although intermediates did not necessarily have the wealth of experience of the experts, I still needed to interview them to get more in-depth responses about their various modelling experiences. I therefore used the same interview questions used with the experts in the field to gather data from this group, which would allow me to compare and contrast responses from the two groups. Question one was modified to get an idea of their modelling background, and for half of the intermediate participants a list of heuristics was given to aid in answering question seven. We will discuss the implications of this, as well as an analysis of all responses in Chapter 5.

## 3.5 Group 3: Complete Novices

To complete the spectrum from novice to expert I first chose a set of participants who were clear novices on the Dreyfus scale: having no skill, needing rules, lacking self-confidence, interested in completing as opposed to learning, and progressing by relying only on rules. These students came from two separate FAN X99 classes held at SFU. The FAN X99 class is a foundations of numeracy course. In other words, these participants had issues with all aspects of mathematics, not only mathematical modelling, with many of them not doing high school mathematics past grade 10. For these students, mathematical modelling consisted of solving word problems. This is the most basic of modelling problems and is what Briggs calls “modelling or story problems” [16]. The work I did with these complete novices was influenced by the work of Schoenfeld and Liljedahl, who both observed and worked with novices in classroom settings.

Forty-five students from the two classes agreed to complete questionnaires for the study. These students had a wide variety of barriers to mathematics, but one common theme among the majority was a fear or dislike of mathematics. This made them an ideal contrast to experts when addressing the affective components of modelling.

The students were accommodating and many of them were willing to discuss their thought processes with me to help me see how they went about solving their word problems. This allowed me to develop a relationship with them, and also gave me a first hand view of the issues and techniques unique to this level of expertise.

The data for this group consisted of responses to questionnaires distributed at the end of the semester. Students were again reminded that they were under no obligation to participate. I chose questionnaires instead of interviews here because I was asking them to relate a specific experience. This was in contrast with the experts and intermediates where there was the need to probe into varied experiences which a questionnaire might not cover. The use of questionnaires yielded a large number of responses (45 students responded). The questionnaire contained some questions that paralleled the expert and intermediate interview. Other questions were included to establish novice students' preconceptions. We will discuss and analyse these responses in Chapter 7.

## 3.6 Group 4: Novice Modellers

Although the FAN X99 students qualify as complete novices, they are not expected to progress along the spectrum to modelling expertise in this course, as modelling is not the primary focus of the course. (This is not to say that the data from the FAN X99 students was useless, far from it, as we see in Chapter 7.) This led me to SFU's fourth year undergraduate modelling course Math 461, in which students are encouraged to participate in the Mathematics Contest in Modelling (MCM). Although this was a fourth year course, many of the students had very little or no modelling experience. This made them an appropriate group of novice modellers, as their lack of experience qualified them as novices, but their decision to take the class and participate in the MCM showed a likelihood of moving along the spectrum towards expertise in modelling.

Those who were participating in the MCM were asked via email to fill out a questionnaire about this experience. The contest is a weekend long modelling competition, in which students work together in groups of three on a given problem. Students have a choice of two problems to work on and are not allowed to consult anyone outside of their group for help. Eight of the MCM participants agreed to complete the questionnaire (after a second reminder a month after the MCM), with one of them not being a member of the Math 461 class.

The data for this group of novice modellers are their responses to the questionnaires. This questionnaire was different from that for the acute novice. Students were asked to comment on what made their chosen problem a modelling problem, as well as being asked several questions that paralleled the interview of experts and intermediates. Due to the lengthy turnaround in responses some students explained that they could not remember clearly all the details of the MCM. The responses collected will be discussed further in Chapter 6.

## 3.7 Analysis

The data in this study are the interview and questionnaire responses. While no discourse analysis was done, pauses and exclamations that highlight a particular point were included. The observations were used as a backdrop to the analysis of the transcribed and questionnaire data. As the data is primarily spoken words, it was edited to allow the reader to follow the train of thought of the speaker.

After the expert and intermediate interview data was recorded and transcribed, I transferred the data to an excel spreadsheet. This allowed comparison by question as well as by person. Individual responses to each question were then coded using line-by-line coding informed by Grounded Theory [48]. To do so I looked at each line in a given response to a question, and summarized it. I then looked at these lined summaries and identified any themes of interest within them. I noted recurring themes as well as outliers and compared the results across groups, contrasting the responses of the intermediates with those of the experts. In many cases new codes had to be created specifically for the intermediates.

While the coding was done using the principles found in Grounded Theory, the analysis of the themes was not. Charmaz [48] describes Grounded Theory as developing theory as it emerges from the data, thus the theory comes from the data as opposed to the data being analysed using existing theories. In place of this method of analysis I made use of Patton's [49] principle of analytic induction. Sriraman [50] explains that Patton's principle works well when studying "an extremely complex construct involving a wide range of interacting behaviours." Since the literature had motivated my study, common themes that emerged were compared to the existing literature using Patton's principles, as opposed to developing a theory from the ground up as Grounded Theory suggests. This is because " 'analytic induction, in contrast to grounded theory, begins with an analyst's deduced propositions or theory-derived hypothesis and is a procedure for verifying theories and propositions based on qualitative data.' (Taylor and Bogdan, 1984, p.127)" [50].

Questionnaire responses for both groups of novices were also transferred to spreadsheets to allow comparison by question and by person. These responses were succinct and did not require coding. Responses of the MCM student-novices were compared and contrasted with those of the experts and intermediates. For the FAN students I focused more on their preconceptions of mathematics. However I contrasted them with the experts to highlight the differences here, and in some cases, the similarities. In the following chapters we look at the detailed analysis of responses for each group.

# Chapter 4

# Experts

The experts were the first group of participants to be interviewed. These were professors, all known in the field of mathematical modelling, and many of them currently teaching modelling courses. Many of these professors work at the some of most prestigious universities in the world including Oxford and UCLA. My primary objective in interviewing these individuals was to understand how experts view modelling, specifically trying to understand their definition of modelling, how they feel when they approach a modelling problem and their tendencies when becoming stuck while working on a model.

As explained in Section 3.3 many of these experts participate in week-long study groups hosted by many different universities and societies. While in these study groups, the experts worked with those from industry, colleagues and students in order to find solutions to industrial problems. These study groups were referenced in several responses to the interview questions.

In this chapter I will discuss the results of the interviews of the experts, presenting a few illustrative quotes highlighting the main themes raised by the experts. Note that the names Bob, Ted, Tony and Mike refer throughout this study to experts who chose to remain anonymous.

## 4.1 Defining Mathematical Modelling

**Question 1: How do you define Mathematical Modelling?**

The definition of mathematical modelling is not a clear-cut entity. As expected, among the experts in the field we note several variations in their definitions; however, three common themes of interest emerged. The experts generally agree that mathematical modelling is a *description* of a *real life situation* using a *mathematical framework*:

> *"Description of a physical problem, in terms of mathematical equations." (Brian Wetton)*

> *"Something that takes things in the real world and describes some sort of mathematics to them. A precise mathematical problem." (Andrea Bertozzi)*

> *"It is essential to clearly state your assumptions and only once that is accomplished can one move forward to build a model that attempts to capture the essence of a physical process. We're usually dealing with a physical problem, but mathematical modelling is broader than this. Industrial mathematics focuses on that branch of modelling that is primarily concerned with problems of either a societal or commercial benefit. One of the largest challenges in this field is the clarification of the problem. Typically researchers are presented with irrelevant information and understanding the essence of a process is more valued than its exact quantification. Abstracting the problem allows one to make connection between physical problems that on the surface seem completely unrelated but using mathematics peels back this obscurity and allows for cross pollination of ideas. This deepens the understanding and allows for innovation." (Sean Bohun)*

For Sean Bohun understanding the problem is an important aspect of modelling. This theme shall be addressed several more times throughout this chapter and beyond. For three other experts, it was highlighted that the model would be a simplified or approximate version of the physical system:

> *"A process by which people take a situation in the real world and put it into the simplest possible mathematics (at least at first) that describes the process. And by describes I mean captures the effects or experience. This is different from analysis or verification." (Bob)*

> *"It's taking a real world problem coming out of some other scientific area and trying to formulate mathematical equations to describe it to some level of approximation or other." (Thomas Witelski)*

Four experts expand from here, expressing modelling as a process, thus solving, analyzing and verifying the model are also parts of the definition. These experts then go on to talk about refining the model for more accurate results or using the model to make predictions:

> *"Mathematical modelling begins with a problem in the real world, which has principles, structure and cause that we have an understanding of. Mathematical modelling takes us from the physical world to a model via formulation, mathematization and idealization. If we could get a perfect model it would contain every detail, but there is no perfect model. Instead we may get a hierarchy of models where we take out more and more details for the sake of simplification. Each model is self-contained. The model then goes through the process of solution/analysis/computation and we assess it to see if we identify model behaviours that are analogous to real world behaviours. If not we need to go back to our model and refine it. This process is what mathematical modelling is." (David Muraki)*

This creates an interesting dichotomy on what mathematical modelling is. Is it exclusively the setting up of the model that defines it or do the solution and use of the model need to be considered as well? This dichotomy may affect the teaching of mathematical modelling: one group might mainly focus on setting up the model "from scratch", while another group might focus on continuing past the initial set up to solving the model. Being able to set up the model is considered more an art than a science, and in some ways more difficult to teach as will be discussed later (see Sections 4.4 and 5.4).

A third and completely different response given by one expert is that mathematical modelling does not need a definition, as it is no different from doing any type of problem solving:

> *"I don't think it needs a definition really. People just pretend it's something which is different. I don't really think it's any different to anybody who works in any particular subject you know? I mean you just do it. Everyone does it if they have a problem." (Colin Atkinson)*

Again we would expect that this view of modelling would have an impact on how it is taught. Is modelling exactly the same as problem solving? I believe there are important distinctions in the context of tertiary education, which will emerge from this study, and will be discussed in subsequent chapters.

## 4.2 Initial Thoughts, Plans and Feelings

**Question 2: Upon seeing a problem, what are your first thoughts, feelings or plans?**

**Feelings:**

One of the things that is often debilitating to novices are the overwhelming negative feelings associated with working on problems [5, 13]. For this reason, I was interested in knowing what feelings the experts experience when first faced with a modelling problem. The first point of interest is that almost all of the experts had to be prompted twice to answer this part of the question. They were forthcoming with their initial thoughts and plans, but had to be reminded about the fact that the question asked for feelings as well.

After prompting, several of the experts spoke of positive feelings: excitement, interest and curiosity:

> *" 'Ooh what's going on there?' or 'I have no interest in this whatsoever' " (Bob)*

> *"Depends on the problem. Is it a good problem? Is it interesting? Do I think the solution will make me a better person? Or the world a better place? Is it a problem I can solve? (There are lots of problems that I like but I can't solve, because I'm not equipped to solve them.)" (Lou Rossi)*

> *"Interviewer: You've touched on when you see a problem what your first thoughts and plans would be but you haven't said what your first feelings would be when you see a problem.*
> *Brian Wetton: Well, feelings! (chuckles) well. So I don't know if I'm answering your question but there are certainly some things that get me more excited than others. There's all kinds of mathematically interesting questions for example in math finance, but finance just doesn't do it for me. I've worked in modelling*

> *hydrogen fuel cells, so that kind of physics, it's especially interesting because it's multiple phenomena that are coupled together in a complex way, that's the kind of thing that gets me excited. So if I see something that's really interesting my feeling is one of excitement trying to see how you would describe that in mathematical terms." (Brian Wetton)*

These experts illustrate the desire for the problem to be of interest to them. They are not interested in all problems, and do not work on those they do not like. Lou Rossi further highlights the need for the problem to provide him with purpose, supporting the work of Pink [41]. These feelings of interest and excitement when modelling are expected, as their expertise in the field is correlated to positive feelings towards their field of choice [45]. Even more interesting, is that seven of the experts interviewed spoke of negative feelings: feeling daunted, fear, panic, and wondering if they could solve the problem. Indeed there was some overlap and three of them spoke of feeling worried and curious.

> *"Oh my first feelings! I usually feel very daunted, doing some thing new. Yes it's usually something new for me and I usually feel like: wow I have a whole lot that I need to learn here. But I also feel excited I should say because it's always exciting to tackle a new problem." (Tony)*

> *"Feelings, it's kind of a joke but, 'I hate this problem for being hard. I hate myself for not being smart enough to know immediately how to write it down,' or some variation." (Thomas Witelski)*

> *"Well, feelings! That doesn't help! How am I going understand this? I have no idea what the words mean. I don't know anything about this." (Sam Howison)*

Three experts of these seven went on to explain that they were able to ignore, deal with or suppress those negative emotions.

> *"I've learnt to suppress research anxiety years ago." (David Muraki)*

> *"I think you're worried and curious yeah. But you have to deal with it. You have to begin things that's how you progress" (Colin Atkinson)*

> *"Not knowing how to do it is a motivator. I'm fearless. I don't mind not knowing what's going on and I'm willing to argue until understanding is agreed. There are two types of students those who want to apply those things they know or those who want to learn something new." (Sean Bohun)*

The fact that they all first ignored their negative feelings or recognized them but were able to push past them is consistent with many of the findings in psychology, both in positive psychology [37] and work on mindset [38] where resilience is identified as a key to mastery.

**Thoughts and Plans:**

All but two of the experts explain that their first plans were to explore the problem and do research or gather data.

> *"Explore it and see where it takes me." (Burt Tilley)*

> *"Oh you mean when I first see a problem that I hope to model? My first thought is that before I even think about mathematics that I should try to learn as much about the particular problem as I possibly can. If it's a problem in biology I try to read biology papers and talk to biologists or anyone else who's working on the problem and really just find out about the biology in the problem. And I think that's crucial in order to be able to accurately narrow it down into a mathematical description. So that's the first thing I think, 'cause I don't even think about math, I just think I had better learn about the background." (Tony)*

> *"Just learn about a subject. You have a problem you want to learn about it right? Learn about it, try to make sensible decisions about it and if you're a mathematician you apply the tools of your trade to it." (Colin Atkinson)*

Additionally three other key themes were discussed: understanding the dominant process, simplifying the problem and collaboration.

Understanding the dominant process:

> *"Depending on the specifics of the problem, the path to understanding is to first identify the broad strokes of what is happening, i.e., the dominant process. To*

> *this end, are there similar problems I have worked on? Do I need to review the underlying theory? Is a preliminary literature search necessary? These are always questions that I ask myself. Occasionally I will choose a problem at a workshop simply because I know the least about this material and I need to learn more. The unforeseen benefit is that by doing this you add a fresh unbiased viewpoint." (Sean Bohun)*

> *"Mostly when I first see a problem I try to think: what area of science do I need to go to figure out how to write it down mathematically? Is it a physics problem? Then I have Newton's laws. If it's a chemistry reaction problem then I use the chemical reaction kinetics equations, mass action. If it's from some other area can I use geometry? Or how can I break it down into simplest pieces that make up the problem." (Thomas Witelski)*

Simplifying the problem:

> *"I suppose my first thought would be: what's the simplest example of whatever it is that someone's showing me. If it looks really complicated, is there an easier way to look at it, or, not even an easier way to look at it, is there an easier problem? Maybe I can start with that one. The easier the better." (Mike)*

Collaboration:

> *"If you want to take modelling seriously, if you want to be able to come up with a model in a given situation, then [...] you do need a repertoire of things, which [...] you acquire with a bit of experience. I'm a big advocate of going to meetings like the [...] PIMS workshop and the Canadian study group meeting. Where they have these open problems from industry [...]. A load of people from industry come with open problems that they would like to see solved. And you have no idea what you're going to get. And then a bunch of mathematicians sit around and spend a week [working on them]" (Sam Howison)*

> *"Well, you usually start with a discussion with someone who's an expert in the application field. It's usually in that discussion where they highlight what they at least believe are the important phenomena and in that discussion you (well I*

*think what you should do, but which sometimes means the problem is not mathematically interesting) try and think of the simplest mathematical description of what they're saying. What, the simplest that it can possibly be [...] I think one of the useful things a modeller can contribute to a scientific project is asking the questions: is this important? What does this do? Because it really forces the application people to think about really what is going on. They tend to get sometimes sidetracked with particular descriptions of the problem and you can come in and say,'Well we could include that additional effect that you always neglect, it's not hard and maybe it's important.' " (Brian Wetton)*

These above-mentioned themes connect strongly with Polya's work on problem solving. For Polya, step 1 is "Understanding the Problem," which we see re-iterated by Lou Rossi:

*"I feel better if at the end I understand the problem better. I want to understand it in a better way." (Lou Rossi)*

The vast majority of experts interviewed identified understanding the problem as their first step, whether by research and exploration, discussion or simplification. The key observation here is that all the experts begin with the assumption that they do not even understand the problem. This, as we will see, is in direct contrast with the complete novices who are not even aware that they do not understand the question asked, implying poor metacognition [35, 51]. Lacking awareness of one's own inadequate understanding demonstrates the so-called 'Dunning-Kruger effect' where "people [who] are incompetent in the strategies they adopt to achieve success and satisfaction, [...] suffer a dual burden: Not only do they reach erroneous conclusions and make unfortunate choices, but their incompetence robs them of the ability to realize it." [51].

## 4.3 Dealing with Initial Barriers

**Question 3: If you have no idea how to start what do you do?**

The most prominent theme to emerge was trying to better understand the problem before starting. Eight of the fourteen experts do this by discussing or collaborating with others:

*"Talk to other people. Mathematical modelling is not a solitary activity. Ask someone who you think knows [about it]. Read a [research] paper." (Bob)*

> *"Talk to other people. They may not know how to do it, or be able to tell you. But just knocking around ideas. It's a reflective process." (Lou Rossi)*

> *"If it's a person I'm talking to then I probably try to get more information from them. If I don't know where to start then I might ask them, 'Is there a simpler problem that's related? I don't really know how to do this one right away but, what if there wasn't an $x^2$ term? Or what if there wasn't diffusion?' If I'm having a conversation then I would try to say, 'What can we do to make this problem look like something that I might know how to begin?' " (Mike)*

We notice here that in order to understand the problem better Mike raises the theme of simplification. This may be through a discussion on how to simplify, but others speak of simplification as a means of getting past being stuck initially, without necessarily collaborating with others:

> *"Try to solve a related problem, a simpler related problem is better. Solve the trivial problems in different regimes. Try to sneak up on the problem." (Sean Bohun)*

> *"[7 second pause] I try to take a small part, a really small, it could be just a microscopic part of the problem. So usually a problem comes with a goal: the person proposing the problem says, "I want to find the answer to this thing about the system." And so if that seems overwhelmingly difficult then I'll start ignoring his question and just try to answer something I think I can. And then maybe using that as a first step for getting to something more difficult. But it's usually just, first build up your own confidence that you understand what's happening with at least parts of the problem." (Thomas Witelski)*

> *"[5 second pause] Not recently actually. Because really you should start with the simplest thing and that's usually pretty straightforward. Now it's certainly true that that doesn't always work. So usually you get stumped later on when the simplest thing doesn't work, doesn't describe what's going on. There's something more complicated because what you're doing just doesn't match the experimental data that's in the literature." (Brian Wetton)*

Seven of the responses indicated that research and looking at others' previous work was a method for getting unstuck initially. This again ties directly to Polya's first heuristic *"Understanding the Problem:"*

> *"Try to see if someone else had done it. Collaboration. Make sure you have an understanding of the resources at your disposal. This comes from experience and practice" (Burt Tilley)*

> *"If I have to solve it and I don't know where to start I look around and see if someone else has solved it." (Lou Rossi)*

> *"Then you have to study more [laughs]" (Ted)*

Burt Tilley's comment about understanding resources points to Csikszentmihalyi's statement that creative individuals make do with whatever resources are at hand to reach their goals [45]. Apart from this two experts advise taking a break or waiting: "Wait it out" (David Muraki) or "sleep over it" (Reinhard Illner). Three others experts advise the opposite course of action, just starting: "You have to start somewhere" (Colin Atkinson), "just start writing" (Lou Rossi) and "you've got to start trying things" (Sam Howison). Sam Howison goes on to give a full description on how he would solve a problem he knows nothing about:

> *"So the problem might be something to do with data packets in a mobile phone network, which is something I do not claim to know something about at all. Though you could start saying, 'Where do they go and what do I need to know in order to describe the system?' I'd need to know how many packets there were at this place and how many there were at that place. Did I need to know how many there were in between? Then I'd ask the guy proposing the problem and see if the guy says, 'Yes, you do need to know,' or 'No, you don't need to know because they go at the speed of light,' and that kind of thing. And then what are the rules whereby they get moved around from one place to another? Say someone takes a look and says, 'No, that one's full, send it somewhere else,' start writing down little equations that say how these things are conserved, if they are conserved. They may disappear. But assuming they are conserved (which I guess probably is one of the aims of mobile phone people) you add up and you balance equations.*

*What goes in must come out again, or it must stay in the system, so you do all of that kind of thing. You write down a great big discrete model or continuous model if you're dealing with a continuous system. If you're going to do that it's very useful to have basic knowledge of the difference between things like a conservation law that says that mass or momentum or energy or something of that sort is conserved, and a constitutive law which says how does a particular material relate to types of stress or strain or temperature or thermal energy, temperature gradient and heat flux all of those things. So that you have a good idea what is allowed in any case and what you need to know in addition specific to the system you're talking about." (Sam Howison)*

One strikingly different response was the expert who explained:

*"I'm not usually in that situation, usually I'm working on a problem because I have some ideas about it. I'm not usually in a situation where somebody says, 'You have to do this,' because I wouldn't do that. I work on something where I have some ideas." (Andrea Bertozzi)*

This statement fully correlates with David Muraki's views on why modelling is more stressful for the student than the expert:

*"Of course you can imagine the situation where someone says, 'We know David Muraki is an expert in this field and we've been waiting for him to comment.' Now someone has set the bar for me. For students it's implied that the person asking has expectations about what they should know, so this is very stress-inducing. Additionally there's a lack of confidence novices have that they can make intelligent responses, and this is near panic-inducing." (David Muraki)*

Finally Colin Atkinson brings up an important point in his discussion below about why you get stuck at the beginning of a problem.

*"The hard part is formulating it really. There are two aspects one is formulation the other is solution." (Colin Atkinson)*

Once again we see that modelling can be split into two steps as discussed in question 1 (Section 4.1), but here Colin Atkinson brings our attention to the fact that in starting a mathematical modelling problem the hardest part is formulating it. This step is not at all straightforward and leads us to our next question.

## 4.4 Determining Relevance

**Question 4: How do you determine what information is relevant to the model?**

For me this was a question to which the answer was not immediately apparent. In my personal experience with modelling, trying to capture the entire phenomenon exactly was very difficult, would often create unsolvable equations, and I was unsure how one decides what information is relevant to the model. This difficulty was echoed by Sam Howison and Mike who refer to this process as being an "art".

Of course, this question assumes that not all of the information available is necessarily relevant to the model. We are assuming that we should "try to include the fewest possible phenomena that will describe it" (Bob). Six other responses similarly address using the simplest model possible, reinforcing the theme that was raised in question 3 above (Section 4.3):

> *"If I were to say what were the precepts in modelling, one of them is: always do the simplest problem you can first, always solve the easiest, simplest problem first." (Sam Howison)*

> *"There are some simplified facts that are central to the model, so the crucial modelling step is to be able to distil from the series of observations those relevant central steps and throw away the things that you can add on later that's icing on the cake. But that's difficult, and so the way we normally [proceed] now is not to try to explain everything, every detail, but to have the simplest model that explains the basic operation of what you observe. It's understood that a lot of features that you won't be able to explain you need to ignore them for a while." (Ted)*

> *"Because the type of modelling that I do, my broad interest is really in pattern formation, I'm really interested in understanding basic mechanisms of pattern formation. So the models I tend to build are really quite minimal. For me the best model is the simplest one which is real, which is reasonable and which reproduces the phenomena of interest. So I will always start with the most simple models I can dream of, before going to something more complicated." (Tony)*

Notice that the simplified model does not need to replicate all results exactly or “try to explain everything”. It is enough that the simple model “reproduces the phenomena of interest” or “captures trends” qualitatively. But how exactly do we go about finding this simplest model? Seven experts explained that the data or physical properties of the problem help to determine relevance:

> *“To build a model we need to mathematize the real world principles and structure. Hence, we need to understand the world and have the appropriate mathematical vocabulary. You have to know the reality you’re talking about to talk about it mathematically.” (David Muraki)*

> *“Again this depends on the problem. Usually it is clear from the outset. Usually the phenomenon is explained to you and they want an explanation. I try to embed the problem into language and concepts I know, (e.g.: kinectic theory).” (Reinhard Illner)*

> *“Ok. How do I determine what information is relevant? Well I think modelling is always a process of iterative refinement. So if I take a few iterations I really understand what is relevant. So I guess I create a model which gives some results. We try to think about those and compare them to whatever data might be known about the system that were trying to describe. And then the model might well need to be changed.” (Tony)*

> *“What can you find out, in other words don’t start making theories before you have a good idea of what actually happened, is a very good idea it’s always very easy to come along with your own particular branch of expertise, fluid mechanics as it were, and to rephrase the whole problem in terms of what you know how to do. But first you need to look at the data and just see what’s going on. So if people don’t do that then they will end up in trouble, that’s for sure. So that’s what, if I was going to have cardinal rules in modelling that would certainly be one of them.” (Sam Howison)*

The theme of the language of mathematical modelling has been raised here by David Muraki and Reinhard Illner, as understanding the language is important for understanding the problem itself. A more prevailing theme presented itself of using the data to help determine what is relevant, be it experimental or field data. Brian Wetton also observed that seeing how the experiment is done can give insight into what would be relevant to the model. Sam Howison goes on to introduce the concept of using the conservation laws, which also depend on the physical properties of the problem, as we are trying to conserve those properties.

This ties in to the four responses on understanding, or seeing what is relevant. For Reinhard Illner, the way to help understand the problem is to rephrase it in his area of expertise. Sam Howison warns that while rephrasing it in your area of expertise is easy you must be careful to look at the data and confirm what is actually happening.

Five experts speak about using the mathematical technique of non-dimensionalising the problem and looking for the significant parameters this way:

> *"But people that are starting, you don't always know how to [simplify]. I mean you won't have that experience so you should keep everything to begin with. And then you can, by doing dimensional analysis and taking ratios of the different effects you'll get numbers that are either big or small. And those parameters, those ratios will tell you that this effect is much more important than that effect so I can neglect that term compared to this term. Dimensional analysis is [...] a very simple sounding mathematical approach but it's one of the most useful guides in modelling for knowing which things to consider." (Thomas Witelski)*

> *"Which of course requires that you have an idea what simplification is and that means you do have to have some notion of such things as asymptotics and the ideas of scaling and non-dimensionalisation so that you can have a guess at what are the big mechanisms and what are the small ones in a given situation." (Sam Howison)*

> *"Hopefully we have access to an expert and are interacting with someone who's not a mathematician. Working with them you can deduce what's important. Usually you can reduce things down. Reducing non-dimensional quantities (which is basically what applied mathematics was in the 80's). You have to be careful though, not to always assume a parameter is not important because it never has been in previous models." (Lou Rossi)*

Lou Rossi cautions us here about prejudging a parameter as unimportant based on previous experience.

Four experts re-introduce the theme of talking to others and collaborating. Four others explain that this skill of determining relevance is one that develops with experience:

> *"Develop a 'spider sense' by having a go." (Bob)*

> *"So that's something that improves with experience. The people that I've seen that are even more experienced than me can do that very quickly almost off the top of their head because they've solved like dozens of a similar kind of problem before." (Thomas Witelski)*

Two experts here also explain that sometimes you start with a more complicated model, and then move to the simpler one from there, as opposed to starting with the simplest one to begin with:

> *"By getting as much information as you can. Make sure you know the whole problem. The equations will tell you what to get rid of. Two ways to model are start with the simple model and add things, or start with the complicated model and remove things, by using the physical properties. You can also use a quick and dirty solution, but don't lose the big picture." (Sean Bohun)*

> *"But all of that assumes that first you've written down the biggest nastiest possible version of the problem that has everything included in it. That's always the safe starting point, if you're not sure put it in the equation and we'll cross out the things that will turn out to be small later. You can't start with something simpler [as] it's harder to kind of put back things unless you've started with the whole thing in the beginning." (Thomas Witelski)*

In many ways, these responses are similar to the previous ones except that they perform some of the first details on paper rather than mentally, perhaps even subconsciously.

## 4.5 Skills Needed for Modelling

**Question 5: What knowledge, skills, and techniques do you think help you to do Mathematical Modelling successfully?**

This question was split into two parts. I explained to the interviewees that while I believed there were certainly mathematical content skills needed to model, I also was sure that there were other skills, life skills or characteristics of personality that aided in successful modelling well. This line of questioning offered a rich set of responses.

The experts were able to list various topics in mathematics, which aided them in modelling. A majority (nine of fourteen) explained that calculus, and partial or ordinary differential equations were useful to them: a consequence of the fact that these experts generally work in continuum modelling. Apart from calculus, six experts each mentioned probability and statistics, and computing or numerical analysis lending to their ability to model. Several other topics were mentioned including: linear algebra, abstract algebra, analysis, data analysis, queuing theory, graph theory, discrete mathematics, calculus of variations, and optimization.

Even more interesting were the responses given to the non-mathematical category. Once again the theme of simplification was mentioned, which is not surprising as it is ubiquitous in the literature. A majority of experts also raised a couple new themes. Eight of the experts explained that breadth as opposed to specific math knowledge was important:

> *"They must also be able to recognize things (the mathematics) in different contexts." (Burt Tilley)*

> *"Being able to couch problems in many different ways. Usually the mathematics is not hard, finding the right mathematics is!" (Lou Rossi)*

> *"I think the ability to recognize the affordances and limitations of different mathematical modelling frameworks first of all. I think you have to have a good knowledge of what you put in and what you get out of different broad classes of models so that you can choose between those things." (Tony)*

> *"The other side that I think is really crucial, is to have a fairly broad based understanding of mathematics and I think this is something that comes from experience. The more experience that one has in different types of problems, and different mathematical techniques, the more success one might have with the problem. Experience is really important." (Andrea Bertozzi)*

> *"Not specific content knowledge but I think [...] how an expert organises [his/her] content knowledge is going to be pretty crucial. [...] So I'm thinking specifically about when I learn new things I'm trying to sort of fit them into what I already*

> *know as opposed to [thinking] it's just another random fact. It's like fitting some kind of structure. I think that's an important thing, not just what one knows but how one organises it." (Mike)*

These experts agree that while some specific content knowledge is helpful, a breadth of knowledge is more useful in developing models. The ability to recognize the mathematics in several contexts is additionally identifies as an indispensable tool. Mike reinforces the need for meta-cognitive skills seen in the literature [18], by explaining that all the knowledge needs to be organized in such a way that it is easily accessible. This ties in with the seven responses endorsing relevant knowledge and the six responses that explained that a background in sciences, particularly the area from which the problem comes, is highly advantageous:

> *"The ability to read a wide range of scientific literature. Because like I said, to do a good model of something in biology or chemistry, you have to actually understand something about biology or chemistry. Definitely [you have to have] the ability to be able to read a range of scientific literature." (Tony)*

> *"Let me not answer for myself but for someone who [would like to be] successful in modelling. That person needs to know that discipline well. A common pitfall, for people doing mathematical modelling on the mathematical side, is to assume that since we know all the advanced mathematics we know more than people that are in the field, and if I just apply my dynamical systems to the biology I'll be able to explain everything. That usually is not true. Someone has said that a mathematical modeller or applied mathematician working on a particular problem in an area should be indistinguishable from the person and the practitioners of that application area. You need to know everything and understand all the data. But you have all these mathematical tools and you have an ability to distil, to generalize, and that's what you apply to what you see in the field." (Ted)*

The assumption above is that the modeller will be working with experts from the field, which explains why the modeller must understand the background of the problem. This also implies that the modeller must be able to communicate and collaborate. For eight of the experts interviewed, the theme of collaboration and communication reappears here:

> *"Also I think the ability to communicate verbally with scientists from fields of application. Not just reading the papers but actually talking to them and having interchange with them. Both of those things are crucial." (Tony)*

> *"I think it's really important to be a good communicator and be able to talk with people who work on the application side. Find out what it is they know, what they don't know and to try to put what they know in the mathematics. You have to be able to ask the right questions and understand the problem from different points of view." (Andrea Bertozzi)*

> *"Maybe the most unusual thing I can mention is perseverance in terms of group interaction, because usually these things happen with a group of people. And sometimes you may have to have some political or diplomatic ability in working with the group to convince them that your approach is worth pushing through." (Thomas Witelski)*

> *"Oh what life skills? Well I'll tell you two of them right away. One of them is persistence, and the other one is listening. You will never make a successful modeller unless you listen to what the experimental people or the industrial people or your collaborators [say]. You're going to be making models in another discipline. It's going to be in industry or it's going to be in another scientific discipline. And if you don't listen to what they say, if you aren't prepared to go the extra mile and learn what the words mean and what's going on, then you haven't got a hope. So the sort of person who walks in and says, 'Give me the equations I'm going to go away and analyse them mathematically,' is not doing mathematical modelling. They may be doing mathematics (and maybe good mathematics) but it's not going to be mathematical modelling." (Sam Howison)*

This reinforces the idea that mathematical modelling is a group activity for many of them. Many collaborative skills are mentioned here: the ability to ask questions, the ability to listen, being able to get results via communication, and being diplomatic. This is one aspect that separates modelling from traditional problem solving which is taught as a solitary activity. In particular Sam Howison explicitly states "learn what the words mean." This is something that novices do not understand: some words mean different things to different people. This is one reason for missteps from the less experienced – they think they understand what is being said when they don't (Chapters 6 and 7).

A variety of personal qualities were listed as being useful to the modeller: maturity, confidence, passion, curiosity, flexibility, stamina, persistence, hard work and patience, among others:

> *"They also need a mathematical maturity which helps them to really understand the content knowledge." (Burt Tilley)*

> *"Ok a third skill is patience. You need patience to know that your first six models may be thrown out for one reason or another. So you should be prepared to go round the loop of: make a model, analyze it as far as possible, compute it, look at the results, compare the results with the experiment or whatever information you have, find they don't fit, try to work out what made them not fit, and then do it all over again. You may have to do that many, many times, so patience is certainly a virtue. I suppose also tolerance of your colleagues, because modelling on the whole is a team activity rather than a solo one. So you do need to be able to get on with people and have a beer with them afterwards." (Sam Howison)*

> *"As for sort of other skills that I would draw upon, I suppose sometimes just [...] the ability to go out on a limb and not really worry about getting it wrong. And maybe it's stupid but we'll figure that out. And if it is then, 'Whatever, I tried it'. So I think a confidence to try things even though I might get it wrong, I'm still going to throw it out there and try it. And maybe someone laughs, maybe it's wrong, maybe I feel like an idiot but hey, I tried it." (Mike)*

These responses illustrate that many different personal characteristics and content knowledge outside of mathematics, help a modeller become successful.

## 4.6 Dealing with Being Stuck

**Question 6: What do you do when you are stuck partway through a modelling problem?**

This question was one of the most important: as a student and a teacher, how do we help ourselves and our students become unstuck? I did not assume that as experts they would claim never to be stuck, but instead that their expertise would allow them to move from this state often enough that modelling was not a frustrating experience for them.

This assumption was supported by Lou Rossi and Andrea Bertozzi explicitly stating that being stuck is a regular occurrence for them. Of particular interest is the fact that Andrea Bertozzi explained in response to question 3 (Section 4.3) that she is never in the situation of being stuck at the beginning of a problem, because she would not choose to work on such a problem. This illustrates that even though an expert has the ability to decide which problem he/she is working on initially, this does not mean that they will never be stuck when working on the problem. Unfortunately, this is something that students typically do not see in class. Through example, it appears to novices that experts always know what to do.

So what do the experts do when stuck? The theme of communication reappears here. Six experts recommend talking to others and either: go back to those who brought the question, talk to others with whom they are working, or sometimes talk to people who are not involved in the project at all! What is important is that collaborating often is a useful skill when stuck on a problem:

> *"Go back for more! More data, interrogate people more." (Lou Rossi)*

> *"Oh I usually talk to people. I mean that's usually the first thing I try. So if I really have been stuck and I'm unable to unstick myself to the best of my ability, then I'll try to talk to people about the model. And that could be talk to other mathematicians if I'm stuck somewhere mathematically or it could be to talk to scientists from whom the data I'm interested in comes from. Talk to them to see if there's something I've missed." (Tony)*

> *"Andrea Bertozzi: I would say a non-trivial percent of the time I go talk to, other mathematicians. One of the things that I do if I have a problem where we're stuck, and I think that the problem may have something to do with the math, I talk to people who may be experts in parts of the math that I'm not. That has been incredibly helpful on several occasions. If I need to do a piece that I don't have the real expertise to do, I go and find somebody who does, and it's good for me because I usually learn something as well.*

> *Interviewer: you said you talk to other mathematicians; do you ever talk to the people who've brought the problem to you?*
>
> *Andrea Bertozzi: Of course, but that's part of the standard procedure I would say. I wouldn't call that getting stuck." (Andrea Bertozzi)*
>
> *"The other thing which is very difficult is, so you are stuck on the math. It's math you don't know. Your model has some structure that you want to get a handle on. One thing that came up with us is that one of the hardest things is [when] you don't even know what it's called. It's a math field, maybe. And maybe there's hundreds of people working on it with beautiful results that you could use, but you can't even google it because you don't actually know what it's called! Because you just came up with it from the application. I don't know what you do then [laughs]. You know this is hard though right? Then (this is very old fashioned): you talk about it in various places and you ask people in the audience if they've seen anything like it. But you give 3 or 4 talks a year, that's not the speed of development that you get with google if you knew what the name was." (Brian Wetton)*

Brian Wetton raises an important point here. Research can be faster and can often point you to those you need to have a discussion with; however, it is difficult to find the right people to talk to if you are not sure what mathematical field your problem falls under. Two responses point to research as opposed talking to others in order to gather more data:

> *"I look at the literature, you know, what others have done." (Reinhard Illner)*

This theme was not widely expressed and was outweighed by the number of responses on talking to others instead.

Simplifying the problem is another theme that re-emerges. Experts recommended this theme if one was stuck initially. Similarly, three experts explain that this approach is one they would try when stuck in the middle. Note that Lou Rossi also recommends complicating the model, as it may be oversimplified:

> *"I look for simple examples, as simple as I can make them to learn more about the structure. And often you discover new features about your problem that help you along." (Reinhard Illner)*

> *"If it's a mathematical hang-up that I'm having, that tells me that maybe I need to simplify the model. Maybe I need to let go of some of my modelling details and simplify the model even further." (Tony)*

> *"Simplify it more. Or make it more complicated (maybe you don't have in enough information to solve it)." (Lou Rossi)*

Many experts speak of doing something different when stuck in the middle of a modelling problem. This includes: thinking differently, trying a different approach, doing a numerical exploration, reconsidering assumptions, changing the experiment or model and even starting over:

> *"[14 second pause] I guess it depends on how [...] you define stuck or what led you to be stuck. But often trying a different approach: if you were doing things analytically, it might help to just say ok whatever equation you have, instead of doing it on the board or on paper and pencil let's compute it. Let's put it onto matlab or let's write a simple program, something quick and efficient to see: [were] our calculations that we were doing analytically right? Or did we make a mistake someplace in the middle of the board. So getting confirmation using a slightly different approach can often help cut through things that may be serious roadblocks because you've led yourself to a contradiction. Sometimes you need to back up and then start with a different way to get to where you think you're going." (Thomas Witelski)*

> *"And sometimes I play with the computer, I may use maple or matlab in order to do some numerical exploration." (Reinhard Illner)*

> *"The problem [of] getting stuck, [could be that] your original assumption is wrong and your model does not explain the essential things. Or the prediction of your model contradicts [...] some of the essential behaviours of what you're trying to model. Then you should be prepared to change the model go back to the drawing board and start all over again." (Ted)*

Three experts go even further and explain that they wait and think or do something else completely, which is qualitatively different from trying a different approach:

> *"Do another problem for about a week. Let the ideas come to you as opposed to searching for them." (Burt Tilley)*

> *"You do have to give yourself time, you have to let your brain do the work, which means you have to be thinking about the problem, but you would have to let your subconscious do the work. So you have a shower, you have a bath, go for a walk, go swimming, beer whatever. And when you come back to it you may well find that things are coming, that your brain has sorted [it] out" (Sam Howison)*

This idea of your subconscious doing the work while you pursue another activity completely speaks directly to Hadamard's work on incubation and illumination in mathematical creativity. However, another expert explains:

> *"Creating new mathematics is not part of what mathematical modellers do." (Bob)*

This appears to be in direct contrast to waiting for the inspiration of a new idea. Bob's belief also explains why eight other experts seek help from other sources, whether it be from people or literature, to see what has been done before.

The common theme here however, is that these experts all seem to assume they are capable of solving the problem, or that a solution exists already. They are simply missing some critical insight or perspective. Their experience has led them to discuss with others, or change their thinking, or simplify in order to gain this insight. But in some cases, they need to take the time to let their subconscious arrive at the solution. They also do not simply say, "keep trying" as some novices do (see Section 7.6), but each have specific and often multiple strategies for dealing with being stuck.

## 4.7 Heuristics

**Question 7: What heuristics do you use most often? (e.g. draw a picture, work backwards, exploiting a related/simpler problem etc.)**

In his work *Ants, Bikes and Clocks*, Briggs speaks of some twelve different heuristics for problem solving. Knowing that the experts would have their own rules of thumb for solving problems, only a few of Briggs' heuristics were mentioned as a method of providing context.

The two most popular heuristics used by the experts were drawing a picture and exploiting a simpler problem. Many of them mentioned that they were visual people and needed a picture of some kind in order to better understand the problem.

> *"I draw a lot of pictures. I really think very geometrically usually. But this is a very individual thing. I also like to teach that way, I teach with a lot of graphics." (Reinhard Illner)*

> *"Drawing a picture and using the simplest problem to start with are good." (Thomas Witelski)*

> *"Always draw a picture. Physical intuition is a bit of a two-edged sword because it could be wrong, but drawing analogues with other situations." (Sam Howison)*

> *"Well definitely draw a picture. Probably I wouldn't even know what the person's trying to tell me if there's not some picture. So if someone else isn't drawing it, I'm going draw it and say, 'Is this what you mean?' 'Is it like this?' 'Are you looking at it from the side?' " (Mike)*

The recurring theme of simplifying the problem and trying to solve the simplest model should not make this heuristic a surprise:

> *"I would say one thing I do try to do, I try to always boil it down to the simplest mathematical model that captures the important parts of the problem. If you put too many parameters and factors into the model it may be too much for the model. So keeping it as simple as possible is really important. If we end up with a model that we think is correct physically, but we don't know how to solve, another thing we do is come up with a simpler model that has some of the features that we can solve. That's where knowing a lot of detailed mathematics helps, because you can say, ' I don't know how to solve this really hard problem mathematically, but I think I can solve this simpler version,' and you go from there. You do it in stages." (Andrea Bertozzi)*

Apart from these two main heuristics, exploiting a related problem was mentioned four times and the theme of communication or talking to others three times:

> *"Solving a similar problem: the joy of maths is the ability for different circumstances to generate the same maths." (Bob)*

> *"Talk to application people. Write down some simple things. Look at experiments. Both how they're conducted and what comes out." (Brian Wetton)*

> *"And asking questions is the other big, big heuristic. If there's someone, an experimental person, ask them some questions, because you'll develop your own intuition of what's going on." (Sam Howison)*

One other technique mentioned by two experts was looking at limiting cases and understanding the behaviour there:

> *"[T]his is something we commonly do in applied mathematics [...] I try to see if limiting cases of my model make sense. So if I make some parameters in my model very large or very small, or if I make my system size very large or very small, do I get the behaviour that I expect to and does it jive with what would be seen or what has been seen in nature? One heuristic I use is to look at limiting behaviour." (Tony)*

> *"I'll tell you a really useful heuristic which is in the what if line: ask what if one of the parameters becomes very large or very small what would happen? Get a really simple limit where you can actually see what would happen, and then does that tell you anything about what might be happening in the more complicated limit? So don't just always say all the parameters are 1 and 2 and 3 and so on. Say, 'What happens if I make that one large or that one very small?' It may not tell you anything but it's always worth trying. It's broadening the physical universe a bit to consider physical set-ups, which aren't exactly the same as the one that you're thinking about. So trying a few thought experiments is always a good idea." (Sam Howison)*

The experts highlighted two of the heuristics mentioned in the question: drawing a picture and exploiting a related/ simpler problem. We have seen the theme of collaboration raised before as well. Interestingly, a new theme was identified: using a limiting case. This is the only one of Briggs' twelve heuristics that was raised independently.

## 4.8 Verifying Solutions

**Question 8a): How do you check that you are pursuing the correct solution?**

This question underwent a change as early responses such as "correct solution means?" (Bob) highlighted the fact that in modelling "there is no correct solution" (David Muraki). This prompted me to start asking, "How do you check that you are pursuing a consistent or sensible solution?" to which ten experts responded that comparing with the data or the experiment was the way in which to do this. Four of these ten experts simply speak of comparing results to data.

> *"Asking the person that proposed the problem, 'Do you have any data that we could use as a guide to validate or invalidate what we've done so far? Something to check?' " (Thomas Witelski)*

> *"I picked up a flyer somewhere in the bank or out of my mailbox where some company had published monthly payments for a mortgage under certain terms. Of course this is an example that didn't require any new research, because it's all available. But I basically set the task to myself and to the students to say: now apply our theory and verify that this is correct. And it was really rather interesting because while the first part of the theory gave us the correct figure within a few cents, it didn't give exactly the same figure. And it turned out that I had overlooked a small legal detail that exists in Canada, that regulates what kinds of published rates companies may use. Anyway after we took this into account it came to perfection. So essentially the short answer is I try to check with the data that is available." (Reinhard Illner)*

Two of the experts did not specify what form this data might take, and another one gave examples of the data being a graph, a simulation or something observed. Reinhard Illner indicates above that his data is information that is readily available or came from a file and possible experiment. He also highlights that the model should be internally consistent.

Four of these ten experts stated that their data comes from experiments. Specifically their work comes from modelling some given experiment, and so a comparison to the experimental results helps verify the accuracy of the solution:

> *"You always want to go back and compare to experiments. That might suggest redesigning the experiment as well or looking at a certain aspect of the data. When you develop the model that would be what the physicists would call theory, then you always go back and compare your theory to the actual experiment." (Andrea Bertozzi)*

> *"[Try] to deduce predictions and then see how they compare with experiment or the intuition of the people that know what's going on (or your intuition about what's going on if it's something that you know about.)" (Sam Howison)*

> *"You're guided by experimental work and maybe you have to use some of the experimental results to fit some coefficients. So then of course you always worry: have you used the data too much [...]? So the standard thing that they do is, even if you have all the data at the beginning, you try and separate it. So you use some of it to fit and you see if it matches the other. That's a nice sign right? When you have a model you need some coefficients often to match what's going on, but then it can match something else. If you can't do that, then you ask them to do some other experiments afterwards. Maybe you have an idea that it will make things go better (whatever better is). If they do something slightly different than what they're doing now, you get them to try it out. If it does do better, then this is the utility test of the model, that you've predicted something new or you've helped their optimisation process. I mean that is why people want to model, is because it's cheaper and safer than doing experimental work." (Brian Wetton)*

Andrea Bertozzi explains that the experiment may have to be reproduced or redesigned. Sam Howison suggests comparison of the predictions of the model with the experiment, and is the only one who mentions intuition. Brian Wetton cautions that we have to be careful with the experimental data: the data that you use to create the model should not be the data that you use to test it.

Finally, two of the ten experts spoke of comparing results to reality or nature, as opposed to results discovered in a lab:

> *"All my problems are usually motivated by some observation in nature, [so] I compare what I'm getting to nature." (Tony)*

This is due to the fact that their work is based on modelling phenomena occurring in nature as opposed to modelling work done by experimentalists.

Alternate ways to check the "faithfulness of the model" (David Muraki) were mentioned. Other ways include checking that the model makes sense, checking that it works as expected or looking at the qualitative behaviour of the model. One could also see if something fundamental can be explained. Three experts touched on looking at limiting cases which was introduced in response to the previous question (Section 4.7):

> *"Come up with a reduced problem. Simplify the problem initially, solve that problem first and then check if the complicated solution matches the simple one with the correct parameter set to 0." (Burt Tilley)*

> *"There're the obvious ones: is the answer positive when it should be? But that's just to eliminate mathematical errors. There's always the check that you ask: is the solution doing what it should? And at any stage: are there any special cases? Which is again what I was saying just before, let that parameter be 0 or infinity and then look at special cases. Are there any special cases where you can see what's going on that means that your theory is not wrong? This doesn't mean it's right, just means it's not wrong." (Sam Howison)*

Four experts mentioned the accuracy of predictions, as another way to compare to the data. This re-iterates the dichotomy observed in answer to question 1 (Section 4.1) that for some experts, predictions are a part of the modelling process. Ted speaks of the simplest problem that can make predictions that are verifiable:

> *"There's this hypothesis called Occam's Razor that you need to come up with the simplest explanation for a problem. If your simplest explanation not only answers the question of that particular problem but actually could make predictions on problems that were not asked, but then once you ask them can be verified, then you know that you probably are on the right track" (Ted)*

Two experts touched on what to do if there is no data with which to compare. In this case they compare the solutions to two different methods to see if they are consistent:

> *"Well, I always use numerical simulations or some independent approach to kind of validate an analytical thing. Or I've always looked at least two independent ways to solve something one analytical one numerical." (Thomas Witelski)*

> *"I also always try to, I usually for all my projects have some numerical and some analytical results. My work is usually a combination of those things. So I make sure that those things are also in agreement with each other and in support of each other." (Tony)*

In summary, these responses show a willingness by experts to stretch the situation under consideration beyond reality by assuming symmetry, limiting parameter values or problem reduction to test the model. As we will see in Chapters 6 and 7 novices instead are typically bound by the constraints of the given parameters and configuration. This means that they tend to proceed with less confidence.

## 4.9 Changing Strategies

**Question 8b): What makes you change solution strategies?**

If you checked and realized you weren't getting a sensible solution, then switching strategies would seem to be the right thing to do. With this in mind, creating this question 8b) as opposed to a new question seemed appropriate. Two main themes emerged from this question, clarifying my idea with a bit more detail. Changing solution strategies can happen as a result of being stuck or getting unsatisfactory results.

Five experts explained that being stuck was a reason for changing solution strategies.

> *"Hitting a wall would be one. But I think, at least if it's a group effort, we're probably going to be trying a bunch of solution strategies at the same time anyway, so if one of them isn't really progressing anywhere then I might sort of leave it on hold and think about something else." (Mike)*

Mike also explained that trying several solution strategies at the same time is the norm when doing modelling in a group. This is not unique to Mike, as I also observed this happening whenever working in modelling camps. The group tends to split tasks and approach problems in several different ways concurrently, ensuring that all solutions are coherent and compatible. While Mike likens being stuck to hitting a wall, Lou Rossi uses a more poetic description of changing strategies in this situation:

> *"It's a lot like water flowing downhill. If a method is working really well and*

> *going along nicely, we like that. When it slows down, tributaries form and I follow the path of least resistance." (Lou Rossi)*

As expected, poor results are the main reason for switching solution strategies, with eleven experts speaking to this point. Ted re-iterates Brian Wetton's caveat on using the same data to create and test the model (Section 4.8):

> *"Suppose you come up with a theory and equation, whose solution explains what you are asked to model. As I mentioned the next step then is to make predictions that were not part of the original problem that was given to you. You make predictions and someone else goes out and measures those, and compares with you, and if that contradicts what you have, then you need to re-examine what you have. See that's the danger of having a theory that explains everything that was given to you [if you] have no more observations for it to verify. So the verification has to come from observations that you don't know and nobody knows and it's not part of the input to your model. Once you have the model and you set to make predictions that will be independent verification." (Ted)*

Other experts speak to poor results without going into such detail on what data was used to test the model:

> *"If things in reality aren't being captured then the model needs to be rethought." (David Muraki)*

> *"Well obviously if all the predictions were completely at variance with the evidence. You would then have to go back and look rather carefully at everything you'd done." (Sam Howison)*

> *"Well, if I get poor results. [laughs] That happens a lot you know, that you make a first assumption, you try it and it doesn't do what you expect. And that [means] you have probably made wrong modelling assumptions." (Reinhard Illner)*

> *"If I'm trying to do analytical types of proofs or if I'm doing parameter explorations within a numerical simulation [and] I cannot reproduce the behaviour that motivated the model building in the first place, that's a big clue to me that something's wrong." (Tony)*

Notice that these experts expect to get a working model that provides a solution; however they also appear to be very aware that the solution may not match the data. This often results in a review of their original modelling assumptions.

Finally Andrea Bertozzi speaks to both issues: being stuck and unsatisfactory results:

> *"If it didn't agree with the data. You know the obvious things right? Or you can't solve it. Nothing that would be a surprise there." (Andrea Bertozzi)*

However apart from these "obvious things," there are two reasons for switching strategies that were each mentioned by one expert only. The first is the idea that the complicatedness of the problem would prompt Sam Howison to switch strategies.

> *"Another thing that'd make you change your mind is if it's clear that the problem you're formulating is horribly difficult. You might then go back and say, 'Well look, if you really want to do this then we are going to have to solve the Navier-Stokes equations in this horrible, complicated geometry. Are you sure this is what you really want us to do?' So that's also a part of my philosophy of trying to do the simplest problem first. Which is a philosophy I inherited it from John Ockendon who is one of the gurus of study groups." (Sam Howison)*

The second unique reason is whether it is worth the effort: Brian Wetton explains that switching strategies is not always worthwhile based on the results it will yield and the time commitment required:

> *"Oh well let's suppose you come to one of these things where you are not agreeing with what's happening and you realise that the most likely thing is a term that will completely change the structure of your model. Well, then I guess it depends on your timeline, the right thing to do is then you say well, 'Start from scratch and let's put that term in, which completely changes everything and let's see what happens.' This is all at the stage where the kind of models we're talking about are things you could code up in matlab over a week or so. So a week of your time is a lot to lose, but it's conceivable. When you get to the stage when it's a year of your time to make a change then, well then you have to think if it's worth your time to do. Worth it because the application is so important or because you're so interested in it or someone's paying you a lot of money or whatever." (Brian Wetton)*

These two responses point to a desire to do the simplest thing, whether it is in terms of the complicatedness of the problem or the time and effort required in solving the model. Sam Howison speaks of a philosophy of doing the simplest problem first, which he accredits to modelling group originator John Ockendon. This ties in with Ted's discussion on Occam's Razor in response to the previous question (Section 4.8) and Lou Rossi's description of following the path of least resistance.

## 4.10 Difficult vs Easy

**Question 9: What makes a modelling problem difficult/ easy?**

My expectation for this question was that the experts would speak to the structure of the problem being a primary indicator of its level of difficulty. Once again the experts were able to answer this question with enough variety as to show the richness of their understanding of what it entails, raising four major themes: mathematical difficulty, problem familiarity, problem clarity and problem complexity.

The first theme that emerges was that of the difficulty of the mathematics. This was sometimes specifically mentioned in relation to students:

> *"Hidden depth. A problem may appear easy on the surface. For students, lack of math knowledge hampers. Also: lack of insight, lack of maturity and the inability to think in a non-linear way makes a problem difficult." (Burt Tilley)*

> *"For students: complicated equations for those who don't want to use computers, or analysis for those who only want to use computers, makes the problem difficult." (Bob)*

In some instances, the experts spoke about the difficulty of the mathematics for themselves:

> *"It could be difficult because you understand exactly the problem and when you go to try to model it you come up across some difficult mathematics. [...]And similarly for easy [...] maybe even the problem is sort of hard to describe, but once you put it down it turns into the heat equation. It turns out to be mathematically easy to deal with the model itself." (Mike)*

> *"Any problem that appears not to offer any analytical simplification at all, it has to be done numerically, are not my cup of tea, I don't go into that." (Sam Howison)*

Mike describes a contrast between the model appearing easy (because the phenomenon is easily understood and explained) and the actual math being difficult and vice-versa. This correlates with Bob's explanation that "hidden depth" can make problems difficult. Sam Howison describes a problem that requires numerical analysis, which is not his "cup of tea". This suggests more an issue with the structure of the problem in that it does not "offer any analytical simplification at all," rather than the actual difficulty of the mathematics. Brian Wetton also discusses how the mathematics being easy can make the problem an easy one:

> *"Well that's sort of tricky. 'Cause it can be easy in a number of ways. It can just be very simple type of mathematical equation that describes the phenomenon accurately. In which case you write your code over the weekend and then you fit some parameters and then you're done. It can actually be quite a complicated phenomenon but very similar to something that there has been a lot of development in. So if it's anything that is really governed by some kind of fluid flow of a particular kind, that an industrial computational fluid dynamics code could solve for you then, even though that could be geometrically very complex and very delicate solvers are needed, but those are all in place. So it could be really hard but if it fits into one of these packages then you might be ok too." (Brian Wetton)*

For Brian Wetton the mathematics being easy is mentioned in conjunction with the fact that the easy mathematical model accurately describes the phenomenon. Secondly, he discusses the theme of familiarity with the problem, the subject area or the solution methods. Six other experts also address this theme:

> *"When you know the answer! That sounds silly but it's what you know. If I'm asked to solve a problem in (for the sake of argument) graph theory, I know almost nothing about graph theory. So I can't tell an easy problem when it walks in the door! But if it's a certain [type of problem then I think] 'Oh that's easy! I know how to do that.' Reformulate it, (a lot of applied math is about reformulating*

> *things) and it becomes such and such problem with this novel twist." (Sam Howison)*

> *"Also, I think it's just important what is the sort of general state of knowledge of the underlying systems. I mean when we model a fluids problem sometimes the modelling is relatively simple because the rules, the basic rules of fluids are so well understood and have been so well agreed upon for so long, that the mathematical description is not controversial. In the field I work in it's less well agreed upon what are the actual underlying rules that govern the system. It's hard to translate something into mathematics when there's a lot of uncertainty about the thing you're trying to translate." (Tony)*

> *"A problem is definitely easy if it fits into one of the standard areas of science. Or it has a strong resemblance to previous problems. So that's the easiest kind of thing you could hope for. But that makes it a little less interesting." (Thomas Witelski)*

Sam Howison explains that a lot of applied mathematics is reformulating, which is not a trivial skill. As an expert he has developed the ability to see one problem and recognize that it can be reformulated into a problem that he is already familiar with.

In terms of attitudes when doing mathematical modelling, we see that Tony chooses to work with the more controversial type of problem rather than one that is well understood and therefore easy. Thomas Witelski expresses here that the easier types of problems for him are not as interesting and Reinhard Illner echoes this mentality:

> *"Well I think it's easy if tools that you're familiar with, that you find readily in the literature apply to it. Then it's easy. It's usually hard if you have to develop the tools yourself in order to make any progress. Tools are the most important aspect of mathematics. Tools are much more important than results, because you can often obtain a result by just using a bunch of recipes. But then the problem is easy. A problem is interesting when the known recipes don't apply and then you have to develop your own toolbox for it." (Reinhard Illner)*

Here Reinhard Illner discusses having to develop your own toolbox or mathematical methods for solving a problem. This is difficult, but also what he deems interesting. Andrea Bertozzi is also familiar with having to solve problems using models or mathematics never done before:

> *"I think from the mathematical standpoint what makes a problem difficult, there are a couple of things: one is if the problem is such that it demands new models that have never been explored before, that would be a challenge. Another challenge is if the model problem involves mathematics that hasn't been done before." (Andrea Bertozzi)*

The differences between some answers: "creating new mathematics is not part of what mathematical modellers do" (Bob) (see Section 4.6) and Bertozzi's above statement: "another challenge is if the model problem involves mathematics that hasn't been done before," arises due to both the extraordinary level of mathematical sophistication of Andrea Bertozzi and the fact that Bob typically works on a very short time scale – one week workshops – rather than year long projects. Andrea Bertozzi goes on to discuss how collaboration has helped her to identify and learn new techniques for solving difficult modelling problems:

> *"We were stuck on a problem and a colleague of mine was [helping us]. In one case it was an expert in conservation laws and in another case it was an expert in probability, statistical physics. And in both cases I actually learned new math by working with them, which was great." (Andrea Bertozzi)*

Once again we notice that the attitude adopted when dealing with the difficult problem is not a negative one and is actually what these experts find interesting. This is also seen in the language that they use. They speak of a problem being challenging and interesting when it is difficult, even if they do not know the tools needed to solve it. This is in contrast to the complete novices who speak of frustration in the face of a difficult problem (Chapter 7).

Problem clarity was the third major theme raised in response to this question. Four experts spoke to this theme:

> *"A problem is difficult if it is very flexible, open or the question is not clear." (Sean Bohun)*

> *"Well I suppose that, one thing that makes it easy is very clear and well defined experimental data and natural observations. If there's clean data, that's much*

*easier to model. When data is not clean I think it's much harder to be able to describe that." (Tony)*

*"Well it could be difficult because it isn't well-posed, and I don't mean that in the mathematical sense. I mean it could be difficult because whoever's presenting it is either not at all familiar with modelling, or just hasn't really thought about how to ask their problem. So if someone's just describing everything. It could be difficult because it's sort of hidden in that sense [...]And similarly for easy, you could walk in and someone's already thought about the problem and they describe it in exactly the terms that you can translate into a model." (Mike)*

Here the difficulty is not necessarily with the mathematics or the modeller's lack of knowledge, but rather with the person presenting the problem. The exception is Tony's discussion about the data being clean. This comes back again to understanding the problem being the first step, as seen in response to question 2 (Section 4.2).

Finally, the theme of the complexity of the problem itself was raised.

*"There are some problems that are inherently obviously complicated. If someone comes along and says, 'I want to make a model for the climate.' Well obviously you can write down some simple balance law models, energy budgets and so on and so forth, but getting any sort of validation, or having any idea about whether their predictions are sensible or not, is clearly going to be a very major task. And if you want to make any realistic models, there are thousands and thousands of variables you should be thinking about." (Sam Howison)*

*"A problem is hard if it has many interacting pieces and the interacting pieces are maybe from different areas of science. Or involve effects that I can't immediately say this one is important and these three other things are not as important, so I just need to consider that one. We need to keep all of the things that are possibly there, then there may be like a parameter for each one of them so I'll have an answer that depends on 20 different parameters. It will be complicated to say anything about the answer except well, here's a program to compute it." (Thomas Witelski)*

*"Hard I think is when the structure of the problem is really different. And that*

> *usually happens if you have a bunch of coupled types of physics that are happening, that's when you get mixes of equations and then the problem's not standard and then it's hard to know what to do and what the math is like, and there's no code you can use." (Brian Wetton)*

> *"All modelling problems nowadays are difficult, the simple problems have already been done. The most difficult problems are the ones where there are more unknown parameters in your model than the observations. And this is a problem for example in say Greenhouse warming, the models that are used have more total parameters than their observations so they always explain what they observed by tuning and yet you don't know whether or not you have the right model." (Ted)*

In each case we see the common theme being many variables or interacting pieces in the model. This complicated type of phenomenon is difficult to model. Sam Howison and Ted give a specific example of climate modelling having this complicated nature.

This discussion was particularly interesting as I expected problem complexity to be the major theme mentioned: however we see that there are several other aspects that make a modelling problem difficult for the experts.

## 4.11 Summary

For the experts there is a dichotomy on the definition of mathematical modelling. For some experts mathematical modelling is a description of the the real world problem, that is, the formulation of a real world problem into a mathematical framework. For others modelling is a process encompassing not only the formulation of the model, but also the solution of that model, verification of the solution, refining and predictions.

Upon first encountering a mathematical model experts focus on understanding the problem, particularly if they are stuck initially. Experts collaborate with the person who brought the problem as well as their colleagues in order to better understand the problem. Experts stated simplification of the problem initially as one of their main heuristics. While prompting was often necessary to get the experts to begin discussing their feelings, several of them spoke of experiencing excitement, curiosity and interest when first faced with a modelling problem. Others spoke of initially feeling worry or anxiety, but were able to move past those feelings to tackle the problem. A question about what makes problems difficult or easy revealed that the difficult problems were the interesting ones for the experts.

The experts interviewed revealed several aspects that go into successful modelling. They have autonomy and choose problems that they are interested in. They deal with being stuck in the middle of modelling by collaborating, simplifying and trying to understand the problem better. They always check that their solution is sensible, usually by comparing it to the data, but also by comparing it to solutions of other methods. They deem several skills important to modelling including a breadth of knowledge and an understanding of the background of the problem. Among the non-cognitive skills valued by the experts are patience, collaborative skills, persistence, maturity and passion.

# Chapter 5

# Intermediates

The second group of participants I worked with was the intermediate modellers, consisting of 8 graduate students and 3 post-doctoral fellows at SFU. The post-doctoral fellows were of particular interest as modelling had been their focus for the majority of their graduate work. While they were in the intermediate range, they were closer to being experts than novices. In this work we shall refer to them as PDF's, when dealing with them specifically in the group of intermediates. This allows a rich landscape of responses and will hopefully provide insight into the transition from intermediate to expert.

Although all of these SFU graduates went through the same graduate program, many of them had completely different experiences due to their existing background knowledge. This, coupled with the fact that most of them partook in the problem solving sessions (see Section 3.4) organized for the applied mathematics graduate students, allowed me to compare how their diverse backgrounds and interests helped them in a common setting.

Using the coded themes of the experts, I noted which of the intermediates identified with those same themes, highlighting the other themes that emerged which were exclusive to the intermediates. Below are their responses to the questions previously asked of the experts. Each intermediate in this study is anonymous, denoted with an initialled 'I.' to mark intermediate status. A few illustrative comments are used to highlight themes of interest.

## 5.1 Defining Mathematical Modelling

**Question 1: How do you define Mathematical Modelling?**

It is noticeable that this description is not as rich as that of the experts. This group of intermediates described mathematical modelling as simply the use of mathematics to solve a real world problem. What particularly stands out is the fact that there is no obvious dichotomy regarding whether the solution is a part of the process of modelling or not. Most intermediates simply assumed that this was the case:

> *"Mathematical modelling, for me it's the defining and solving of problems in the natural or social science, in a mathematical setting, framework." (Frank I.)*

All other themes introduced by the experts here were minimally addressed by the intermediates. Only two intermediates speak of simplifying the problem, another two speak of modelling as describing the real world problem. Two PDF's mention the need to compare solutions with data. It should be noted here that the PDF responses were somewhat richer in their definition of modelling:

> *"I guess it's a way of taking some real life problem and breaking it down into component parts, simplifying it in some ways and then applying known mathematical tools or relationships to try and understand the real world problem and then explore relationships within it." (Tess I.)*

> *"I think mathematics is a language, allowing us to study quantitatively any process in physics or medicine or whatever else. So essentially mathematical modelling is a kind of playing with this grammar with the syntax and so on just to explain better our world. And moreover, we must also check the results with experiments." (Saul I.)*

The themes of the language of mathematics, and simplification are seen here for the first time. Notably no intermediates spoke of models requiring refinement or being used to make predictions. This gives rise to the question: why do the intermediates do their modelling? If the purpose of modelling is simply to solve the modelling problem, then the first answer arrived at may be accepted. However, if they are trying to actually solve a problem in real life, then refinement and/or predictions might become more evident to the intermediates as being part of the process.

## 5.2 Initial Thoughts, Plans and Feelings

**Question 2: Upon seeing a modelling problem, what are your first thoughts, feelings or plans?**

As with the experts, the responses to this question are separated into two main sections: feelings, and thoughts or plans. Of note is the fact that fewer intermediates needed prompting about their feelings than their expert counterparts. There are several similarities and differences that are highlighted below.

### Feelings

Only four intermediates needed prompting to discuss their feelings, and two of these were PDF's. This may be indicative of the fact that feelings play a less pertinent role in modelling as we move from being intermediate to expert. This is not to say that the expert does not experience these feelings, but that they are probably more used to or better at moving past those feelings and applying the more cognitive aspects of modelling.

For two intermediates, while their feelings were addressed without prompting, the question seemed surprising to them:

> *"Right. Oh. Emotionally? Emotionally I like that problem-solving aspect." (Virgil I.)*

> *"Felt like emotionally? I guess it depends on the problem right? And how good of an intuition you have about the problem, because if it's an area that you're not sure about, or it's hard to visualise the physical situation or whatever you're modelling, then it would be a little intimidating I think to be asked to come up with a mathematical model for that situation." (Ryan I.)*

Other intermediates addressed the feelings aspect of the question right away without prompting. In many of these cases (eight out of the eleven interviewed) the feelings referred to were negative: mainly a lack of confidence or a feeling of being overwhelmed:

> *"Most of the time my first feeling is, "Oh my God I need to work on this with someone!" Because I always second-guess [...] what my first thoughts are. And if someone else has the same first thoughts as I do then I feel a little more confident going forward." (Isabel I.)*

> *"It depends, sometimes it could feel a little overwhelming. It could seem like this is way too complicated. There can't be a mathematical model for this." (Frank I.)*

> *"My first feelings are: I have no idea what to do! I guess that often I feel that with starting from scratch. I doubt the possibility of actually coming up with a machine in terms of mathematics, because often I see that mathematics tends to be a lot simpler than what's going on in the real world." (Reuben I.)*

This agrees with half the experts who reported negative emotions. However, the experts were able to overcome those negative emotions. When intermediates spoke of being overwhelmed, this emotion stemmed from the complexity of the real world problem. This explains why simplification is the first plan of three intermediates inundated by the complexity of real world problems. This ties in with the theme of simplifying seen in the literature [19] and utilized by the experts as one of their first plans and also one of the main heuristics they implement (Sections 4.2, 4.7). Interestingly none of the intermediates spoke of overcoming this discomfort or ignoring the panic as some experts did.

In comparison to the eight intermediates that spoke of negative feelings upon first seeing a problem, two spoke of positive feelings. It is also noteworthy that one of them (Virgil I. quoted above) seemed surprised about the question, and the other needed prompting in order to express her feelings on the matter:

> *Interviewer: Ok. First things that you felt, you left that out.*
> *Danny I.: First things I felt?*
> *Interviewer: excited, terrified, nothing?*
> *Danny I.: I wasn't terrified. Nothing really. Not, no, no. Maybe, ok. I wouldn't say excited but you know, eager to get going on it, you know, confident that I'd be able to do something but not, not scared that I don't know, you know what complicated model I'm going to have to look up or anything like that.*

The emotional aspect for Danny I. seemed difficult to pinpoint. Both Danny I. and Virgil I. experience positive feelings but require prompting or have difficulty articulating these feelings. This is in direct contrast to those intermediates that spoke clearly of being overwhelmed.

Only one intermediate spoke of indifference then leading possibly to positive or negative emotions:

> *"At the beginning I'm kind of indifferent and either I gain excitement for the problem or I don't. Usually if I can reduce it to some kind of differential equation then I get excited about it." (Linus I.)*

This response is separated from the others because no other intermediates seem so undecided emotionally about doing mathematical modelling, having clear cut feelings which were either negative or positive.

### Thoughts and Plans

There was some overlap between the themes raised by intermediates and experts. However, the intermediates sometimes differed from the experts in terms of viewpoint. Additionally, sometimes the intermediates only briefly touched on major areas of concern for the experts. The theme of exploring and doing research was mentioned by five of the intermediates interviewed. But where these differ from the experts is that they are all speaking of research strictly in terms of the model itself in an effort to see what others have done:

> *"The first plan is exploratory. And you try to cover the breadth, the most number of paths you could take, just think random stuff. Also research. Like, look up what other people have done. That's, I would say for problem solving, that's the number one thing you can do is, you should dedicate the first 10% of your time, at least 10% to just reading." (Virgil I.)*

> *"First of all you would want to try to look at and see if this problem has been sort of looked at before. Because there's no reason reproducing something that people have already done, so doing a thorough search of it." (Linus I.)*

Only one intermediate speaks about general exploration, in a way that was reminiscent of the expert responses. The experts also referred to researching the area from which the problem comes, to get up to speed on their background knowledge, which is similar to seeing what people have done before.

Five intermediates raised the theme of understanding the big picture or the dominant processes. Notice here that there are several different ways that they go about trying to understand the dominant processes:

> *"But I think the first thing you'd try to do is probably you know draw a picture or something and help sort of organise in your mind what it looks like and what are the factors you're going to have to consider when you're coming up with the model of your whatever your problem is." (Ryan I.)*

> *"If you start [by] coding you're only, you're limiting yourself to one model and then you become concerned with the details, those don't matter. That's just like trivial stuff. That's like if I'm hiking from here to Seymour, instead of me thinking about the overall best path, I'm looking at the ground in front of me thinking, 'Should I step over this log or should I step over that log.' That's just implementation." (Virgil I.)*

> *"I try and basically find what is the simplest and most important aspect of the physical problem that I'm trying to model." (Reuben I.)*

> *"Well I think you have to try and think about exactly what it is you're trying to achieve out of doing the model in the first place. So what am I trying to show? I think that in some way informs your starting point." (Tess I.)*

Identified above at the themes of drawing a picture, organising your thoughts, looking at the overall goal, identifying the important variables, thinking about what you are trying to show and finding what is simplest.

The theme of finding the simplest model or simplifying broached by Reuben I., is touched on by four intermediates in total. We noted earlier in this Section that three of the intermediates who spoke of simplification as a plan were overwhelmed by the complexity of modelling real world problems:

> *"Most of the time you're going to need to make some simplifying assumptions" (Frank I.)*

> *"The first things would be drawing a picture. Drawing a picture of the thing you're trying to model. Or if you can't actually draw a picture then, write down with simple words: first we take this thing, and then we're going do this to it and then we're going to try to see what happens after some time. So the simplest parts of the problem that need to be dealt with." (Danny I.)*

> *"With my MSc thesis I worked at an ecology centre. And when I went to work with them, there were no previous models written down for that kind of problem. So it really was starting from base zero. And I asked myself, 'Well what is it we're trying to do? What is it we're trying to model?' And that was very difficult actually, because you've got a real world problem and you're trying to then break it down with some mathematical model. If there isn't already a framework there then that's a lot more difficult, so you start off I think very, very simply and always go to what's the simplest case scenario. For example modelling an island as a dot on a 1-D line. Although it seems ridiculous but you're trying to solve something in the real world by going to something abstract. I think that's where you have to start because that's where you best understand it." (Tess I.)*

Frank I. speaks of making simplifying assumptions "most" of the time, but does not explain what criteria would cause one not to do so. Danny I. discusses the theme of simplification along with drawing a picture. Tess I.'s response is more elaborate: simplification is used when there are no previous models of the problem to apply to your situation. This ties in with the idea that the first plan is researching previous models however Tess I. explains that this is not always possible and in those cases the simplest model is the best way to start.

Again, it was seen that only a few intermediates touched on the other themes raised by the experts. Reuben I. mentioned the fear of not being able to solve the problem at all, and Tess I. was the only other intermediate to address this:

> *" I suppose you're apprehensive because given that you know you have to make so many simplifications when you first start off, you wonder how relevant it will be. You go through all those anxieties of: what am I hoping to achieve? Will I be able to achieve that? And how will it be used? Already you're thinking about possible limitations of any model that you come up with. So there are all these things going through your mind." (Tess I.)*

Only the two PDFs raised the issue of using their knowledge of similar problems or relying on experience as their first plan:

> *"Often when you have a problem then you can adapt already existing models. So with atmospheric science you're trying to predict future states of the atmosphere.*

> *But you know there are existing models so you already have some sort of basic starting point." (Tess I.)*

In the above case Tess I. refers to using known existing models and adapting them, this ties in with experts' discussion on experience and exploiting similar problems (Section 4.2). Two other intermediates raise the topics of deciding what mathematics to use and trying to write equations as their first plan. These two quotes highlight different aspects of applying mathematics first:

> *"The first thing is: is it something you want to approach analytically, numerically? Should we be thinking about how the code should be structured? Should we be thinking about what equations need to be used depending on which situation you're going to be in?" (Isabel I.)*

> *"Mathematical modelling includes defining the problem mathematically, so once you define the problem mathematically it becomes much more tractable. At that point I believe you separate the model from the actual thing you're studying, but the idea is to stay as close as possible." (Frank I.)*

For Isabel I. the point is to determine what mathematics is necessary to solve the model. For Frank I. the aim of the mathematics is simply to set up the model, so that you can then determine how to solve it. This links to the dichotomy experts' in the definition of modelling.

While Isabel I. in her quote about her first feelings explains that she wants to work with others, no other intermediates explicitly state that talking to others is a part of their plans or thoughts. This is in contrast with approximately half of the experts who expressed this as their first plan of action.

## 5.3 Dealing with Initial Barriers

**Question 3: If you have no idea how to start what do you do?**

In response to this question the experts suggested numerous plans of action: collaboration, research, looking at others' previous work, simplifying, starting somewhere, using special cases, drawing a picture and waiting for inspiration (Section 4.3). Apart from the latter three themes, the intermediates agreed with the above suggestions.

### Research

Among the experts, the most popular course of action when unsure how to start, is to collaborate with others. When looking at the replies of the intermediates we note that this is not the case. Nine out of the eleven intermediates interviewed mentioned research as a solution to not being able to start, making this the most popular course of action for the intermediates:

> *"Oh yeah. You mean besides the research part? Once I've researched I still don't know where to start?" (Liv I.)*

> *"I've never had that, because I always have some kind of plan, be it Googling the terms" (Virgil I.)*

These two quotes illustrate how the use of research seems invaluable and automatic for these intermediates. Virgil I. comments that researching means that he always has an idea how to start. For Liv I., research is such an obvious step that she assumes it is done even before being stuck initially. Three of the intermediates, including Virgil I. above, use blind internet searching. This is different from others who use books and papers in their research to help them start modelling, as seen below:

> *"I guess it's time to start reading probably. So look for some references starting with the most basic references I think you can find. Cause if you start looking at the ones that are quite specific you're probably gonna get lost in the details, you know what I mean? So try to find introductory level references that can get you started on it. And then once you feel that you have enough background information, you can start trying to put together a model on your own. But also looking at sort of more advanced references I guess to see if what has been done for that problem" (Ryan I.)*

> *"Depending on, usually you would try to look up at resources and books and like what, so if it's an area that you haven't ever studied before then you'd probably, you'd first want to get like a basis for it, so you would read through skim through books regarding like the standard approaches for starting these things and then take steps from there." (Linus I.)*

These descriptions are more detailed. Rather than a blanket statement about researching, both explain how this research is done. Ryan I. explains that the way to do this is to start with introductory references. Linus I. suggests reading through books to ascertain the standard approaches to your type of problem, as opposed to general background knowledge.

### Others' Previous Work

Three of the intermediates tie researching with the theme of looking at others' previous work:

> *"You can look up things in terms of community like what other professors might have done, Google or whatever. All these different resources you have can take care of that." (Frank I.)*

> *"I look in the literature. I look for somebody else's ideas basically." (Reuben I.)*

This theme of looking at what others have done was raised by six intermediates in total. This is comparative to the four experts who also raised this as a means of dealing with being stuck initially.

### Collaboration

A total of six intermediates mentioned the theme of collaboration. It is notable that in contrast to the experts the majority of them spoke of research first and collaboration after. One PDF speaks of looking at others' previous work in conjunction with collaboration, and is the only one who does not mention research as a means to start solving the problem:

> *"When you're working on a project with other people you have the benefit of being able to ask your supervisor. If it's an M.Sc. thesis or a Ph.D. then you've got a supervisor, you've got other people who've worked on it that you can go and ask. And they say, 'Well such a body did something similar and you might want to look at their work.' So you've always got that as a starting point but I guess the higher up you go then there are fewer reference points or fewer people to rely on. I guess always work in a team!" (Tess I.)*

Tess I. mentions using others' previous work in the context of working with a mentor as well as others who have already worked on the problem. Similarly two other intermediates also speak of looking at previous work in conjunction with collaboration, specifying that they were working in a group and talking to their colleagues about the problem:

> *"If you're in a group then you start talking to other people because usually the combined experience of the group is going to be far higher than just yours. And everybody's bringing something in. So then you start talking about, you have to sort of agree on some form of model to begin with. We can get an agreement on a way of even looking at the problem, then from there, how to go about solving it. You start talking in the group about what kind of similar problems they've solved and what type of approaches they used to solve those problems right?" (Linus I.)*

This is slightly different from two intermediates who specifically refer to speaking to the expert who brought the problem as opposed to mentoring mathematicians or their colleagues:

> *"We didn't completely know what we had to do, but we had a couple of people from industry there, this one person in particular, who was sort of the chief researcher engineering person with this company that handled road maintenance. If we asked him, 'Ok what's the cost of this procedure versus this procedure?' he could tell us that. Or, 'How long is this one going to last compared to this other treatment?' So we were fortunate to have someone who was already sort of an expert in the field to help us with that." (Ryan I.)*

The experts also recommended this theme of speaking to the person in the field who provided the question when stuck initially, but are also willing to talk to anyone. Isabel I. gives a similar explanation of how having the help of many can help start off the modelling process by sharing ideas. There is, however, no specification of who should be consulted:

> *"I try and consult as many people as possible, because I figure the more input I have the better start I'm going have." (Isabel I.)*

### Just Do Something

In two cases the theme of research is mentioned alongside the idea of just trying to do something:

> *"I look in the literature, I look for somebody else's ideas basically. Or I just mess around. I just do stuff that it's not obvious [...] is going to lead to a solution.*

> *But I'm just interested in this tangential thing and maybe this will give me some insight into what's happening. So those are my two strate[gies], I maybe do a little bit too much messing around [laughs]" (Reuben I.)*

As with the theme of collaboration, research is mentioned first, giving the impression that this would be the first tactic to aid when not knowing where to start, before moving on to just trying something.

### Outliers

Virgil I. pronounces that he's "never had that" situation of not knowing where to start, because he can always do research or something simple:

> *"I've never had that, because I always have some kind of plan, be it Googling the terms, or just trying a small example, or trying out the problem manually doing it. I've never been at the point where I had to really seek external help. Yeah I've never been completely like, 'I have no idea.' That's never happened." (Virgil I.)*

The difference between Virgil I. and his colleagues who do experience being stuck initially, is perception: they consider themselves stuck and research, he researches and thus does not consider himself stuck. Danny I. also claims that this is not a situation with which she is familiar:

> *"I don't think so. I think it's usually the opposite, there're too many different ideas, and difficulty with agreeing with which one to start with." (Danny I.)*

Danny I. clarifies that here she is assuming that modelling is taking place in a group. Although collaboration is not explicitly mentioned, it is implied, and notably as a possible detractor to identifying the first step. Finding the group dynamic a barrier is a theme we will also see raised by the novice modellers in Chapter 6.

Intermediates deal with initial barriers first by research. They also look at others' previous work, collaborate and just try something. Only one intermediate mentioned simplification, which the modelling literature and the experts address repeatedly. Another intermediate raises the point that collaboration can have a downside.

## 5.4 Determining Relevance

**Question 4: How do you determine what information is relevant to the model?**

For the experts, collaboration, experience, the physical properties of the problem, dimensional analysis and understanding the problem were five aspects that helped them determine what information was relevant to the model. This question addressed the important theme of acquiring the simplest model. Similarly different intermediates addressed these themes.

### The Simplest Model

Determining what information was relevant to the model for the experts meant identifying the simplest model that captured the effects of the real world problem. Five intermediates agreed, with one of them claiming simplifying was obvious, another saying it was difficult, and a third pausing a significant time before being able to answer:

> *"Decide which is important and which is to drop? [6 second pause] Well to me it's really obvious what's the simplest thing, it's convincing people that certain details are not important at the beginning, and trying to convince them that, 'It's ok, we will add them later. But for now let's start with the main dynamics of the problem' " (Danny I.)*

> *"When you start out you leave out almost everything that is extraneous to your fundamental dynamics of your problem. I guess that's a very difficult question to answer. You need the bare minimum of something that's going to give you some dynamical properties of what you're studying. But you can leave out certain things that may not change the dynamics that much, but may give you a realistic answer." (Tess I.)*

> *"[19 second pause] I guess that comes down to intuition, I mean if you're trying to model a physical system usually you have some idea about what the behaviour is. So you basically want to come up with the simplest set of things that will at least roughly replicate the behaviour that you kind of expect. Whether you're getting to the quantitative comparison or not." (Reuben I.)*

Both Danny I. and Reuben I. pause before answering this question, suggesting some difficulty in doing so, while Tess I. states explicitly that this is a difficult question. Reuben I. and

Tess I. address the theme of capturing the qualitative as opposed to quantitative behaviour. Danny I. and Reuben I. speak about having some intuition or understanding from their knowledge of the problem. The intermediates address this theme of intuition more than the experts interviewed.

For Danny I. the simplest thing to do is “obvious”. She raises the theme of collaboration once again, speaking of the difficulties that can be encountered when other members of the group do not want to begin with the simple model. This is reminiscent of Thomas Witelski’s claim that diplomacy in collaboration is invaluable.

### Understanding the Main Dynamics of the Problem

This theme of understanding the dominant processes, which was a recurring theme among the experts, was only explicitly referred to by one of the intermediates when determining what variables are relevant to the model:

> *“If I’m given a set of information and I have a good grasp on the problem, I have some understanding as to what I need.” (Isabel I.)*

However, we also see this theme being addressed implicitly. Reuben I. speaks of “having some idea” about the behaviour. Tess I. speaks to understanding the main dynamics of the problem. Danny I. also alludes to the fact that capturing the main dynamics is more important, when she explains that all the little details in the coffee modelling problem should be dealt with later.

The theme of understanding is mentioned explicitly in a different context, when Virgil I. speaks of understanding why you’re good at determining what is relevant.

> *“I think that even people who are good at it don’t understand why they’re good at it.” (Virgil I.)*

This again alludes to the fact that this question of relevance is one of the more “difficult” aspects of the modelling process.

### Collaboration

The experts interviewed recommend collaboration with those who brought the modelling problem to help determine what factors are relevant to the model. Two intermediates also mention this theme of collaboration. Frank I. mentions it in a way that is reminiscent of the experts:

> *"I think it's difficult, cause I mean at that point it becomes more of a problem, more of a question for the person who brought up the, the non-mathematical question, you know what I mean? 'Cause someone at some point emitted a question that you then decided to answer using mathematical modelling. But depending on, on that person's needs and what they want, y'know you're gonna be able to remove some details and then add them on later." (Frank I.)*

In contrast Danny I. gives two scenarios, the first one highlights collaboration with colleagues as useful:

> *"Somebody would say, 'Oh how about adding this?' We would immediately agree that that's an excellent idea [and] once we have a working model, we can add that on. Then the person was happy with that and shut up and let us work. [...] Somebody would go up on the board and say, 'Ok, let's, I'll draw the model and let's see if we agree on it.' And if somebody had something to say they would say it, the guy would continue drawing. If somebody wanted to continue on with the model, they would go up there, continue on, and nobody would interrupt them until they're done. And once they're done they might say, 'Oh did you ever consider maybe adding this?' They didn't interrupt each other, they were patient with each other, and they agreed that certain things could be added later that were not immediately important." (Danny I.)*

In this situation Danny I. was working in an industrial math group. The culture of working together on industrial problems was already established and the members of the group were of the same mind where working on the simplest model was concerned. More importantly, their collaborative skills of listening respectively, and diplomacy were well-honed, which ties in with Sam Howison's theme of listening and other collaborative skills when working in groups.

In the second scenario, Danny I. describes how collaboration can sometimes cause difficulty in achieving the simplest model, as there may be differences of opinion about what should be modelled and whether to even deal with the simplest model first:

> *"They wanted to add fluid dynamics models to how the cream mixes. First of all is it cream or milk? I guess he didn't really specify. Somebody decided to go out of their way to ask: is it milk or cream? [sighs] Should we assume that the coefficient of the cream is the same as the coffee because you know cream is a little bit thicker? You know things like that, that once we have a model with some parameters then we can assign values to based on that information, but for now let's not worry about it so much. And the fluid dynamics, my immediate thought would have been: let's just assume that as soon as you pour it, it is immediately mixed. [laughs] But no, there was a big debate about that which lasted half an hour. Which is very frustrating and it didn't get us anywhere." (Danny I.)*

Danny I. is describing work with a group of graduate students in the problem solving session (see Section 3.4). There is not a general understanding or agreement that the simplest model should be the first approach, which is in contrast to Ted's comment that this is understood in the modelling community (Section 4.4).

**Experience**

Experts and intermediates alike value experience in modelling, with four individuals from each group mentioning this topic:

> *"It seems a little bit, well I hope it's a little bit more effective anyway, to take some of the experiences that you've had and try and decide what's important right off the bat. As opposed to just having like this amount of information and going like, 'Okay, go!' " (Isabel I.)*

Experience is necessary especially at this step in the modelling process because it is difficult:

> *"It's not a well-defined process. It's a process that I think comes from experience. I think, this is probably the trickiest part in the whole thing. I think this is the crux of the problem actually, "Ok you made a mathematical problem, what good is it?" We essentially have no rigorous rules to check or test it. And, I think that even people who are good at it don't understand why they're good at it." (Virgil I.)*

Virgil I. admits that trying to determine what is relevant to the model is not a straightforward endeavour. Intermediates describe it as "difficult" and "tricky", and experts call it an "art."

**Data or Physical Properties**

Unlike the experts, the intermediates did not refer to using the data, but three of them did explain that the physical properties of the system would help to illuminate what factors are important:

> *"I mean if you're trying to model a physical system usually you have some idea about what the behaviour is, so you basically want to come up with the simplest set of things that will at least roughly replicate the behaviour that you kind of expect." (Reuben I.)*

This lack of discussion on the use of actual experimental data suggests that either there is a lack of understanding of the significance of the data, or that intermediates tend not to be working with problems posed by industrialists or experimentalists.

**Dimensional Analysis**

One means of determining relevance is via dimensional analysis, which is different from simply varying parameters, as more analysis and less trial and error is used. Saul I. is the only intermediate to mention dimensional analysis as a means of determining what is relevant to the model:

> *"I think technically what I would do is, perform dimensional analysis and try and using some physical quantities I found in the literature try to, to guess which are the terms or the processes that are less important." (Saul I.)*

However four of the intermediates did speak about varying parameters to see the effects, and sometimes explicitly referred to trial and error:

> *"Once [you] have a basic model, you can try experimenting with varying some parameters and seeing if it makes a big difference or not. And if you start, you can assess your results and assess, 'Ok if I'm changing this one thing, trying it*

> *with some different values, are my results changing by very much?' And if not then, 'Ok well that's maybe something that we can (assuming that we've modelled its contribution correctly) then it's something we can kind of ignore.' " (Ryan I.)*

This raises the question of why the intermediates mention trial and error when the experts do not. However this is not so surprising. Trial and error helps to build understanding and intuition for the problem, which the experts already have. While the experts do not speak of intuition, again the intermediates raise this theme:

> *"And how you make those determinations I have no idea. It just comes from intuition." (Virgil I.)*

> *"I guess that comes down to intuition and kind of like, I mean if you're trying to model a physical system usually you have some idea about what the behaviour is" (Reuben I.)*

We once again must ask why this theme of intuition or "making sense" was addressed by the intermediates but never raised by the experts. Intuition is defined as the ability to understand something immediately without need for conscious reasoning. So we can speculate that for the experts the sub-consciousness of their intuition makes it something that they are not aware of enough to discuss. The intermediates are now developing their intuition and are more aware of it. It is also possible that the experts do not accredit their ability to determine what is relevant to the model to a subconscious thought process, but to a very conscious one. They may believe that it is not intuition that helps them, but deliberate conscious thought and an understanding of the problem.

## 5.5 Skills Needed for Modelling

**Question 5: What knowledge, skills, and techniques do you think help you to do Mathematical Modelling successfully?**

As previously mentioned, this question was split into two parts. I was interested in the mathematics that the interviewees found helpful, but also topics outside of mathematics that may be useful to the modeller. For the experts, six main themes emerged. While they did mention content knowledge, in particular calculus and PDEs, they also addressed having a breadth of knowledge, relevant skills, a science background and the ability to collaborate. Experts also mentioned advantageous personality traits they thought essential for modelling successfully. We will look at all three categories of themes for the intermediates, to see how they were addressed.

### Math Content Knowledge

Among the eleven intermediates interviewed, seven of them spoke of calculus or differential equations being helpful in their modelling process, which is reflective of the fact that these students mainly work in continuum modelling. Another major topic of value to the intermediates was numerical or programming skill. Other mathematics topics mentioned were: discrete mathematics, fluid dynamics, Fourier analysis, probability and statistics, image processing, linear algebra and Lagrangian mechanics.

### Non-mathematical Skills

In contrast with the mathematical skills, there were only three main non-mathematical skills that the intermediates valued: breadth and relevant skills, background and scientific knowledge, collaboration.

*Breadth and Relevant Skills*

The two themes of breadth and relevant skills are closely tied. For the intermediates it is more difficult to separate their thoughts on these two topics. The topic of breadth was mentioned by three intermediates explicitly:

> *"Well I mean, I mean [5 sec pause] I mean I'd stop short of saying there's a specific like mathematical tool, because different problems are going to require totally different tools" (Reuben I.)*

Tied to the concept of having a breadth of knowledge (as opposed to specific content knowledge) is the theme of having relevant skills. As modelling is a broad area covering many topics, it is important to be able to identify the relevant skills and utilize them. Another three intermediates therefore discuss the use of relevant skills:

> *"If you're aware of standard models that have been applied in certain areas, and you can apply them, then that certainly helps. Because you look at it and you can say, 'Oh I know what that's going to need, I can just model that with some fluids dynamics equations' and you know what the fluid dynamics equations are, and you know whether they'll be relevant or not. I think that definitely helps." (Tess I.)*

What is needed is not just the actual content knowledge, but also the ability to figure out what to do, and which skills and methods to apply. In order to do this breadth is required. This is also tied strongly to having knowledge in the field that the problem comes from.

*Science/Background Knowledge*

This theme was raised by seven of the eleven intermediates interviewed. For some it was just a passing comment on the need to study other subjects, but for Danny I. knowing enough background knowledge is important in order to research those things that you do not know.

> *"Quickly be able to know enough about the topic and what kinds of things might, you might need on the technical side and be able to find those online." (Danny I.)*

Tess I. gives a different insight on background knowledge. She notes here that different experts in different fields use the same words to mean different things, or vice versa:

> *"Given that a lot of modelling within applications involves working with industry and often non-mathematicians, it sometimes helps if you have a background not necessarily in mathematics, although obviously you do need the basic fundamentals of mathematics. That's one of the biggest obstacles I've found when I've worked with industry. They have their own terminology, they have their own way of thinking about problems and it's very difficult for people with a mathematical background, who also have their own terminology and way of thinking, to adapt. If you can't make those two meet and you don't have any way of understanding the real fundamentals of the problem away from your mathematical terminology then you're really in a lot of difficulty. It helps a lot to be able to try and understand how others look at a problem and how it's different to the way you look at a problem." (Tess I.)*

This idea of understanding what the terminology in the area from which the problem comes has already been mentioned by Sam Howison when he says, "learn what the words mean". It is important to be able to communicate with those people who have brought the problem to you. To do this, you need to understand their background, their way of thinking and speak their language. Tess I. confirms this with her above explanation of how this was an obstacle for her when working in industry.

In contrast to the other six intermediates Livingston I. explains that knowledge outside of mathematics is not useful to him. This comment may be a result of the fact that Livingston I.'s experience with modelling comes from university courses and the MCM.

*Collaboration and Communication*

Working and talking with others is a recurring theme that the experts deem important when doing all aspects of mathematical modelling. Thus, when asked what skills are helpful outside of actual math content knowledge, many experts re-iterated this theme. The intermediates however did not have the same overwhelming response, with only four of them mentioning collaboration, or communication as an important modelling skill:

> *"I think you have to be really willing to collaborate, because I think that you can't possibly know everything there is to know about every subject. And so if you're given a problem, like ok fine, you have a math background but you don't have any background in where the problem's coming from, then you can't, you're not going to be very, you might be able to solve it mathematically but you wont know if like what you're getting makes any sense because you don't have that intuition from the other side of it. And so I think collaboration is huge you need to know that you're not going to be able to do it on your own. And you know the bigger the brain trust the faster it goes." (Isabel I.)*

Isabel I. explains that collaboration is necessary because the field of mathematical modelling is so broad. Liv I. discusses how working with her supervisor often helps her to understand things in greater depth.

> *"I'm always like constantly amazed when I go and see, meet with [my supervisor]. Like I'll come to him I got this problem I understand it I've got pictures for you we can move on. And just in like 20 minutes with me, he'll [say], 'What about*

> *this? What about this? Have you proven this? Can we look at it in this way?' It just opens up so much that I didn't I didn't even consider, I didn't even think to look at, and ultimately I didn't even understand what I had gotten, [or] why I got the results I had gotten." (Liv I.)*

Reuben I. explains that collaboration can help when you're unsure whether modelling a problem is possible, tying collaboration with confidence. Frank I.'s collaboration with his supervisor is successful because they have differing areas of expertise: numeric versus analytic.

### Personal Qualities of a Good Modeller

Looking at the traits raised by the experts, the main personality traits are maturity and experience; confidence and risk-taking; passion, interest and curiosity; and persistence, hard work and patience. These traits were also valued by the intermediates, and so we will deal with each area separately.

*Experience*

While several intermediates implied that experience was important in their answers, two of the intermediates spoke extensively on this topic. Virgil I. speaks of experience in mathematical modelling being like practicing a sport, and developing your brain muscle memory.

> *"I think experience is the biggest. And what I mean is if you're playing any sort of sport, you can't really concentrate on the overall game unless [everything else] is just muscle memory. So for me I just think about things a lot all the time. I'll just be sitting on the bus and some random thought will come in and I begin to explore it. I think about things that are just of interest [...] you're almost training your brain muscle memory. So, you know that you have to check xyz constraints each time you try something, and that keeps you honest [...] I find I just think about those types of things, I'll read about them. When I'm on the Internet I'll see an article, I'll read through it. Some new idea pops out and then I include that idea so my list of, checklist gets refined and modified, and that's why it's kind of a softie science. It's more of an art than a science at this point." (Virgil I.)*

This answer highlights other characteristics along with experience. He speaks of practicing constantly and of his interest in certain areas, helping him to develop the skills to model those areas of mathematics well.

Liv I. notes the difference between her own experience working with first year calculus compared to students she works with. She then contrasts that with her experience working on her graduate studies compared to her supervisor

> *"When I'm in the applied calculus workshop, I've had experience so I can look at the problems and solve them in a couple minutes, because I've seen the way that these problems work out. You pick up on clues along the way. When you have experience you've seen certain clues, certain things happen before, and you know how to deal with them, you've had experience in dealing with these things. Whereas if you see it for the first time you have to go out and learn how to deal with it [...] I'm always constantly amazed when I go and meet with [my supervisor]. I'll come to him, 'I got this problem. I understand it. I've got pictures for you we can move on.' And just in 20 minutes with me, he'll [say], 'What about this? What about this? Have you proven this? Can we look at it in this way?' It just opens up so much that I didn't even consider, I didn't even think to look at, and ultimately I didn't even understand what I had gotten [or] why I got the results I had gotten." (Liv I.)*

This answer touches on collaboration, and she also highlights the theme of seeing the bigger picture because of your experience. This is something she is able to do when trying to help students with first year calculus. Similarly she notices that her supervisor has a much broader view of the problems that she is working on as a graduate student.

*Confidence and Risk-taking*

Four intermediates mention risk-taking from two different points of view: confidence, and fear:

> *"Of course some kind of optimism in the in approaching the in approaching the problem and also I think it could be useful, an approach like this, ok: I start working now I don't I don't care for the moment if I arrive to a solution or not I work, stop." (Saul I.)*

> *"I think one thing that holds a lot of people back is not being able to jump in. When you're unsure of something, you're afraid to jump in. So I think one characteristic would be people who are good modellers are not afraid to make mistakes. And just try something, you have an idea – try something." (Liv I.)*

In conjunction with Saul I.'s optimism is the ability to simply try even when you're unsure of the answer, but this ties in with fear of failure. Liv I. explains that a good modeller cannot be bound by this fear. This is especially true since modelling consists of constant mistakes and failure:

> *"Well you have to be ok with getting it wrong, a lot! For the first little bit you know? Because if you think that you're going to walk into this and you're always going to be on the right track on your first step, you're going to be thoroughly disappointed and walk away. Guaranteed." (Isabel I.)*

*Passion, Interest and Curiosity*

Several experts when talking about their feelings upon seeing a new problem speak of a passion for the subject. Both Linus I. and Virgil I. speak of their passion or interest in the subject aiding them in being able to solve modelling problems.

> *"I'm one of those people when I have get a problem in my head I can't get it out for the life of me, so I succumb to it and I, I have to work on it. A lot of people when they get a math assignment they procrastinate and wait till the last minute to work on it, and I'm one of those people where as soon as I get an assignment I have to look at it. I have to start thinking about these things because in the back of my mind I know I have to solve these problems. So it's very compulsive that I have to start looking at them right away. And in the long run that is very beneficial because then you're not trying to do things last minute. And you enjoy things a lot more when you're not doing things last minute." (Linus I.)*

For Linus I. there is a compulsion to get the work done. He describes himself later as being obsessive. This is not a negative trait for him, as he will work on a problem until it is solved, and hence is often successful. He also speaks of the enjoyment he receives from working on a problem and not having to do it last minute.

Virgil I. does not describe himself as obsessive, but does say that he thinks about things all the time. He also spends a lot of time thinking about the areas that he finds "neat".

*Perseverance and Hard Work*

This was a theme raised by several experts. Sean Bohun spoke of stamina, while Thomas Witelski called it perseverance and Sam Howison called it persistence, hard work and patience. All of these words describe the ability to keep going. For many mathematics students the idea of working for a long time on a problem is daunting. We will see when dealing with the complete novices that perseverance is not attached to solving mathematics for many of them.

Virgil I.'s analogy of sports when speaking about experience highlights that one of the things that he deems important is constant work and practice. One other intermediate recognizes and states explicitly that modelling requires the ability to keep going and persisting through failure:

> *"Chances are you're going to go down the wrong path from some time or another, and if you don't have any perseverance at all you're not going to succeed at all." (Isabel I.)*

The intermediates value several skills when modelling, from calculus to perseverance and hard work. Collaboration is not valued as much by the intermediates as the experts. In contrast intuition is valued more by the intermediates.

## 5.6 Dealing with Being Stuck

**Question 6: What do you do when you are stuck partway through a modelling problem?**

Experts and intermediates alike find themselves stuck when working on modelling problems. Experts Lou Rossi and Andrea Bertozzi explained that being stuck is a regular part of modelling. Two intermediates echo that sentiment.

> *"It's amazing how much you're stuck. Always stuck. Like even I leave [my supervisor], the meeting with [my supervisor] and I'm like, "Ok I know exactly what to do." and I go home and like, literally 30 minutes into work I'm like, 'Oh no! I'm stuck again!' [laughs]" (Liv I.)*

Acknowledging that being stuck is the norm is one thing, but the question is what do you do in this situation. All of the experts had different techniques for getting unstuck. Of the methods mentioned by the experts, the three themes of starting over, simplifying and numerical exploration were not echoed by the intermediates. These omissions are telling, separating the intermediates from their expert counterparts. However the following five themes raised by the experts did re-emerge extensively upon interviewing the intermediates: collaboration; researching and gathering more data; perseverance; doing something else; waiting and thinking. One or two of those interviewed also touched on other themes: reconsidering assumptions; trying a different approach; solving analogous problems; and changing the experiment or model. We will examine the five main theme separately, analyzing the intermediates' responses and contrasting them to those of the experts.

### Communication and Collaboration

When the theme of communication or collaboration appear, there are several different people with which this can happen: with colleagues, with superiors, with experts in the field, and with people who are not working on the problem at all. Three intermediates recommend talking to others even if they are not working specifically on your problem:

> *"I would say talk it over with somebody first and see. My supervisor [...] says sometimes you work on a problem so closely that your nose is only this far away from it and you can't see the whole picture right? So it's good to step back. And I think talking with somebody else, maybe someone who's not as familiar with the problem as you are, can give you insight into maybe something you've missed or some assumption you've made that seemed justified to you but maybe isn't." (Ryan I.)*

> *"In the middle of my Ph.D. there was this certain part of a solve that I was doing and it was giving this error. I didn't understand why or where this error was coming from. This involved a Poisson solve, so I talked to people that worked on Poisson solvers and they said, 'Oh, it's known that sometimes you get this error.' Although they weren't working directly on the problem I was working on, talking to them allowed me to find the source. And I was able to understand that aspect of it [...] talking to people helps a lot even if they aren't working directly in your field." (Tess I.)*

> *"Talking to someone even if they don't know what you're doing, trying to explain it also helps. And for me I use this a lot, I don't know if many people do this, but just rewriting out my notes. Like do I understand what I'm saying, can I write this, can I articulate this out for myself?" (Liv I.)*

Talking with another person can help you gain a different perspective or they may be an expert in a portion of your problem and help you overcome a particular barrier. Talking with others also helps you to articulate your problem, which may in turn help you to solve it. Of course, intermediates also have the option of talking to supervisors:

> *"When I am stuck I do sometimes email [my supervisor]. But, sometimes I don't even email him I just start writing an email to [him]. Sometimes I even just start emailing [him], and I write him, I've written him like 20 line emails or whatever, and then at the end I just don't even send them. Just writing it out or talking it out actually helps me think about the problem and then either I put my finger on the problem, [...] or I know where to look for the problem. Because I'm imagining what [my supervisor] would say. 'Cause sometimes when you're stuck you don't really take a step back and say, 'Why am I stuck? What could possibly give this error or what could possibly give me this problem?' " (Frank I.)*

Liv I. is trying to articulate the problem for herself, and Frank I. for his supervisor, but the results are the same. In trying to explain the problem they realize what it is and are better equipped to solve it and become unstuck. This metacognitive step is exactly what Mason suggests, as writing down why you are stuck helps you to proceed ( [28], p.19). A good supervisor can also provide guidance while the student is still fully engaged:

> *"Well, recently I was working with [my supervisor], and for about 3 weeks he kept telling me that I had to use conditional probability on something. And I conditioned on every possible thing before I conditioned on the right thing. Which was really frustrating because I'd come and he'd be like, 'No, try again.' Because, he's very patient but he does want me to try and get it by myself." (Isabel I.)*

Finally intermediates can collaborate with their colleagues:

> *"Well with the MCM it's different because you have these four days. So it's limited time and also you're usually a team of 3 or 4 so there there's a dynamic of: if you're stuck you can always go talk to the other people, you can sort of bounce ideas off each other." (Frank I.)*

The group dynamic and limited time mean that rather than just communicating or articulating in your head, constant collaboration takes place. Here the collaboration takes place among peers. But as Danny I. explains, lack of experience in a collaborative environment can make this difficult.

### Research

When stuck at the beginning of a modelling problem, nine out of the eleven intermediates interviewed said that research was something they would do to help them get started. So it is not surprising that five of them return to the theme of research when stuck in the middle:

> *"And maybe then, check back in the literature to find some other things that could be useful and moreover, check also your calculations, calculations and your code your numerical code." (Saul I.)*

Saul I. mentions research in conjunction with checking your calculations, which is mentioned by two other intermediates. This theme of checking your work is one that we see in the intermediate responses, but has not been raised by the experts. Checking your work is automatic for the expert (Section 4.8), and so they did not articulate it.

Danny I. describes her experience when working in a modelling camp. Note that here research by one of the members is what helps the group to even identify that they are stuck, as well as to become unstuck:

> *"We were working on something, it was working out, but then somebody, on the second or third day came and said, 'Oh by the way, I did some research and this thing that we did here that is not valid.' But then we all, because we were a lot of us, we actually took some quiet time to think about it and looking up stuff online and then somebody said, "Ok this is what we need to add." (Danny I.)*

This scenario incorporates many themes: collaboration as they are working together in a group; research to identify the problem and to solve it; as well as waiting and thinking.

Finally Ryan I. explains that research is sometimes a difficult option:

> *"I mean obviously there's literature as well you can try to look at that, but that can be like looking for a needle in a haystack sometimes." (Ryan I.)*

This correlates with Brian Wetton's statement (Section 4.10) on the difficulty of online research if you do not know exactly what you are looking for.

**Perseverance**

Persistence was addressed by six intermediates. Isabel I. described (above) her collaborative effort with her supervisor where his guidance caused her to try several different approaches as she was working. Reuben I. explains that the ability to persist and keep trying several different things is one of the reasons he never considers himself stuck:

> *"I guess I never feel like I'm stuck in the middle because when I'm in the middle I've got enough things that I realise that I don't understand [...]. A few months back I was working on this code and some of the tests that I was doing weren't quite working out. Basically I was spending a lot of time like trying to debug it, and trying to figure out what was going on. From [my supervisor]'s perspective I was stuck, but from my perspective I was coming to it every day, I was doing stuff and I was figuring stuff out." (Reuben I.)*

This description re-iterates the theme of checking your work that the intermediates have raised repeatedly. Reuben I. has a different perspective on being stuck: rather than thinking "I'm stuck, I should do these things," as his supervisor believed him to be, he viewed it as, "I still haven't checked all these things yet, so I'm not stuck." This is similar to Virgil I.'s assertion that he's never stuck initially because he can always do research (Section 5.3).

Tess I. gives a completely different example of how persistence can be useful. In this case she also has a difference of opinion with her supervisor about what he perceived to be an error but was not:

> *"My supervisor told me there was a bug in my code and I was adamant that it wasn't a bug. So we had this long process where he kept telling me to find the bug and it turned out that it wasn't a bug. So sometimes, if you think you're right you have to persevere with that. Because otherwise I would have been looking for this erroneous bug forever. Although working with other people can help, that's*

> *an instance of where you also have to trust your own judgement as well." (Tess I.)*

This is an example of collaboration as well as persistence. Tess I. makes the observation that sometimes when collaborating with others it is important not to passively follow what they suggest as it may be incorrect. Linus I. discusses understanding why you are stuck in conjunction with perseverance. He explains why giving up on your model may sometimes be a bad idea:

> *"A lot of times when you get stuck, the solution isn't just to throw your hands in the air and abandon everything. Sometimes there is a subtlety in it that if you think about the problem and you understand why it's not working, then you can fix it. And then it all of a sudden magically works. And at that moment when you're stuck you have no idea that you're actually sort of an inch away from solving the problem. So you need to persevere somewhat, you can't just give up." (Linus I.)*

In understanding why you are stuck, you get a better idea of whether your previous work is useful. This will help you to realize if you only need to persevere or whether or not an entire revamp is necessary. Virgil I. also speaks to the benefits of just pushing through the tough part of the problem. This is an interesting statement because for many complete novices (see Section 7.6) this is the reason they are often not successful. They are simply unaware that sometimes sheer hard work is necessary to get the understanding or the answer. The lack of understanding why they are stuck, as highlighted by Linus I., compounds the issue. They are not even aware when only a few more minutes of work will get them the necessary breakthrough.

### Wait and Think or Do Something Else

From the literature on problem solving we see that while pure hard work is necessary, and persistence is an important skill, sometimes persistence is ineffective. Some modelling experts explain that when they are stuck, they opt to take a break, stop and think or go and do something else completely (Section 4.6). Briggs in his book Ants, Bikes and Clocks describes this strategy as "Don't spin your wheels." However, stopping and thinking is qualitatively different from doing something else completely, as one involves conscious effort and the other involves the subconscious. Three intermediates recommend stopping and thinking, or simply taking a step back:

> *"And the same way you have to, as soon as you get stuck too you also have to start [...] opening your mind to other possibilities. So there could be other approaches that you haven't looked at and so start thinking about what other things that you could possibly do." (Linus I.)*

Another aspect of not spinning your wheels is doing something else. Rather than stopping to think consciously of the problem, doing something else allows the subconscious to work on the problem:

> *"I get pretty obsessive at first. You're stuck [...] you try to hammer through. That almost never works. So you basically, spend two, three hours just doing nothing really. I mean you think about the problem, maybe it's helping but it seems like it's going nowhere. Then I usually think to myself (I mean when I was young I wouldn't, I would just keep going, and that would be useless.)[...]the more you do that the more you get tired and mentally tired. [...] And sometimes I'll work on something else. I'll still be working, but I'll be working on another problem and maybe an easy problem, maybe something I'm more familiar with that might help me change my thoughts. And very often that works. It's surprising. Usually, I'll go to bed and the next morning I'll start working on the problem: within the first half hour I find the bug or whatever there was. Sometimes it's not that easy sometimes it takes weeks but the point is: I try to limit how much time I spend on a particular problem when I get stuck." (Frank I.)*

Frank I. describes almost exactly Hadamard's process of mathematical creativity. In the preparation stage, Hadamard describes the mathematician's work as deliberate conscious effort. This is echoed in Frank I.'s statement on trying to hammer through. Following this is the incubation stage "where the mathematician if stuck in his deliberate efforts he stops working on the problem". This is what Frank I. describes throughout the passage, explaining that he stops working on it and does something else. This leads to illumination "where the unconscious mind solves the problem and the answer becomes available to the conscious mind". Frank I. describes this a little differently, when he says, "within the first half hour I find the bug or whatever it was." Note that this illumination comes with surprise.

**Other themes**

One theme that was raised by an intermediate was that of solving analogous problems. Expert Bob mentions this as a means of getting unstuck. Similarly intermediate Isabel I. has found this to be useful.

> *"There's lots of stuff done on [my area of mathematics] but not in the sense that I am working in it. So I have to try and say, 'Ok this is their problem this is my problem, how do they relate?' " (Isabel I.)*

Isabel I.'s field is one where not a lot of work has been done previously. This makes it difficult to find others' work in her area. Because of this she looks at similar problems to see if they can help her solve her own.

Another theme that emerged was that of change, whether changing assumptions, changing your approach to the problem or changing the model altogether:

> *"So it's good to step back and I think talking with somebody else, maybe someone who's not as familiar with the problem as you are, can give you insight into maybe something you've missed or some assumption you've made that seemed justified to you but maybe isn't." (Ryan I.)*

> *"Check back in the literature to find some other things that could be useful and moreover, check also your calculations and your numerical code. And maybe at the end also check back your assumptions [laughs]." (Saul I.)*

Ryan I. explains that your assumptions may seem justified to you but may not be to another person. For Saul I. everything should be checked, not just your assumptions. Two other intermediates discuss changing your approach to the problem:

> *"So there could be other approaches that you haven't looked at and so start thinking about what other things that you could possibly do." (Linus I.)*

Other than changing the assumptions, or your approach to the problem, the entire model may need to be changed. Experts Andrea Bertozzi, Brian Wetton and Ted all mention that this may be necessary if you find yourself stuck in the middle of the problem, however, no intermediates explicitly raise this theme.

## 5.7 Heuristics

**Question 7: What heuristics do you use most often? (eg. draw a picture, work backwards, exploiting a related/simpler problem etc.)**

For this question I began, as with the experts, by only listing a few heuristics as examples for them to expand on. However I realized that while the experts in the field had several tried and true heuristics that they would use, this might not be the case for the intermediates. I therefore started listing the twelve heuristics mentioned in Briggs' *Ants, Bikes and Clocks*. This gave the intermediates the opportunity to choose which ones they found themselves using most often. The heuristics of drawing a picture, solving a similar/simpler problem or working backwards were mentioned to all interviewed. The other nine heuristics were only mentioned to six of the intermediates. One other heuristic of communication emerged from the interviews. I will discuss the three main heuristics of the intermediates: drawing a picture, solving a simpler/similar problem and taking a break.

### Drawing a picture

Nine of the eleven intermediates interviewed mentioned drawing a picture as an important heuristic:

> *"In terms of when you're just starting out I guess drawing a picture and everything." (Ryan I.)*

> *"I definitely use drawing pictures all the time." (Liv I.)*

Some intermediates simply agreed that drawing a picture was a heuristic used when it was mentioned in the examples. Others went on to elaborate on how drawing a picture can help when modelling.

> *"Pictures will always help too 'cause most people are visual learners, whether they think they are or not. I think it's always going to help because pictures are universal as opposed to math or biology or whatever you're working on. Pictures are more universal. You're going to have a better idea." (Isabel I.)*

> *"Oh I think it's indispensable. I'm a very visual person, so for me I can't even start without a picture. People can describe a problem to me in as many ways*

> *or words as they want, but as soon as the represent it as a picture I think, 'Oh ok now I understand what you're saying.' It helps me understand the problem to draw a picture, and if someone's trying to describe something to me, representing something visually is really, really helpful. I understand people think very differently, not everybody is a visual person but for me personally it helps a lot." (Tess I.)*

> *"Drawing pictures, I've been using a lot of pictures. And, appropriate pictures, [my supervisor] has taught me a lot that you can write a draw a picture you can have a picture but it might not tell you that much." (Liv I.)*

Liv I.'s comment about the pictures being appropriate is very interesting. She explains that you need the right picture for it to be a useful heuristic. Note also that the theme of being a visual thinker or learner emerges. Isabel I. believes that all people are visual learners. This is not a unique point of view. Tess I. adheres to the more traditional view that different people have different styles of thinking. She explains that for her without a picture it is almost impossible for her to understand the problem. This is in contrast to Reuben I. who explains that a physically visual representation is not important for him.

### Solving a Similar/Simpler Problem

For some intermediates, solving a simpler problem was considered the same as solving a similar one. Nine of them mentioned solving a similar or simpler problem or both as being a useful heuristic in mathematical modelling. In some cases this was mentioned simply as "yes" when asked if this was a heuristic they found useful. In other cases this was mentioned as the first heuristic to try:

> *"Simple similar problems, ok the first one is to check a simple model, problems. And also in all of these it could be useful, is to, ok you have your model, you know that in some simpler case that should be, you know the behaviour of the system in some simpler case then check your model in that simpler case." (Saul I.)*

> *"Yeah! Ok, most definitely. So my Ph.D. was a perfect example of this because what I solved was again approximations to atmospheric problems that I could*

> *solve on my laptop really quite simply, whereas if I tried to solve the full problem, with all these extra bits figured in, I wouldn't have got it done. So yes, always try to solve the simplest problem you can. And also, even with your simplest model, you still want to solve that model numerically in a simple way that will give you a fairly accurate solution." (Tess I.)*

This is again in contrast with Reuben I. who states that he would prefer to solve the full problem rather than other problems similar to it:

> *"I think that often that the gap between the simpler problem and the less simple problem is there's a lot of work involved in bridging that. And there are certainly situations in which it's worth it, but often the work involved in bridging that is as much work as it would be to understand the more complicated problem." (Reuben I.)*

**Taking a Break**

This heuristic only mentioned to six of the intermediates interviewed. What is interesting about it is that three of those to whom taking a break was not mentioned explicitly, still said it was an important one for them. They take a break, do something else and wait for ideas to come for them. This heuristic has been discussed extensively in light of Hadamard's work. Here I highlight other ways that taking a break can help in the modelling process.

When asked this question, six of the intermediates referred to taking a break. Reuben I. does not claim to take a break often but notices the value of it:

> *"Oh I'm really bad at that. If I'm stuck I'll just spin my wheels until I'm blue in the face." (Reuben I.)*

Isabel I. speaks about taking a small break. This is not leaving to do something else completely, but a means of stopping herself plunging in before she is ready:

> *"A lot of time I just jump in cause I'm like, 'Oh I can do this.' Nope, nope you can't! Take a step back! So it would be a lot more effective for me if I didn't just jump in with both feet right away." (Isabel I.)*

While this is not the same as waiting for inspiration, the idea of even taking a small break to survey the problem is one that is difficult for complete novices (Chapter 7).

Tess I. explains that taking a break can also stop you from becoming bogged down or sidetracked. She recommends taking a break so that one can stay focused on the big picture:

> *"Yeah I think, take a break cause it's very hard to switch off, when you're involved in something it is very hard to switch off and not get bogged down in [it]. And sometimes you can get sidetracked into exploring things that are really not that relevant. I mean a whole other area could open up and you become obsessed with why is this like this, and in terms of your end aim of what you're trying to achieve in this modelling process it might not be all of that relevant. But you could spend a lot of time trying to explore this issue. It's very hard once your brain gets focused on something to get back on to this bigger picture I think. That's difficult. It helps if you can keep in mind what you're trying to do." (Tess I.)*

To go back to the big picture, this is once again different from Hadamard's incubation theme. Here the break is not to allow inspiration. Instead this taking a break helps her to break away from functional fixedness [52]. She can stop herself from being stuck in a rut and re-focus on what is important for solving the problem.

### Other Heuristics

Two other heuristics of interest are collaboration and working backwards. Collaboration is mentioned several times throughout this work but was only suggested by two of the intermediates as one of their problem solving heuristics. Neither the experts nor intermediates deemed working backwards a useful modelling heuristic, even though it was mentioned explicitly to them. All other heuristics mentioned to the intermediates were considered useful, but only one or two intermediates elaborated on each.

## 5.8 Verifying Solutions

**Question 8a): How do you check that you are pursuing (the correct) a sensible solution?**

The experts all agree that if a solution is found it must be verified, this can be done by: comparing results with the data; looking at special cases; checking that the model makes sense; checking the accuracy of predictions; intuition; comparing different methods. The two main themes for the intermediates were: comparing results with the data and checking that the model makes sense. They also mentioned convergence testing which was not raised by the experts and shall be discussed in this section. Also mentioned by one or two intermediates were the themes of special cases, intuition and comparing two methods.

### Comparing with Data or a Known Solution

Ten of the fourteen experts interviewed raised the theme of comparing the solution from the model to actual data, as a means of verifying your model. This was by far the most popular method for checking the validity of the solution (Section 4.8). Similarly for the intermediates this was the most popular means of checking if the solution was sensible. Four of the intermediates do not speak about comparing the solution to physical data or experimental results. They instead speak of checking their solution against answers they know:

> *"And yeah that's what I would do, just try out a couple things where I know what the answer is and see if my method catches that." (Danny I.)*

> *"So you know there were there were some solutions out there and they all generally gave kind of similar kind of answers but varied in in to certain degrees so I did have some kind of benchmark in knowing that well I should try to re-produce all those results I know that it's going to give me this." (Tess I.)*

This suggests that the solutions which they are using to compare may be known theoretical solutions as opposed to physical data. This is somewhat different from the experts and understandably so. The intermediates are very often still doing "classroom" problems as opposed to real-world problems, and have little access to raw data.

Two other intermediates recognize that physical, experimental data would be useful. Frank I. explains that this keeps him focused on solving the real problem and not being drawn off on a mathematical tangent:

> *"This solution, does this make sense with respect to what I'm modelling, what has been physically observed in a lab? Oftentimes, you always have to try and keep yourself linked to the problem at hand right? 'Cause it's easy to spread out into the mathematical problem and forget about what the actual thing was you were modelling." (Frank I.)*

### Model Makes Sense

When asked how they check that they are pursuing a sensible solution, six of the intermediates interviewed said that they check that the model or solution makes sense. This theme appears to be somewhat redundant but many of them go on to explain what "makes sense" means to them.

Ryan I. explains that the solutions make sense if he can explain them logically:

> *"You should have some intuition of what's a reasonable range of values. And then when you start changing parameters in your model the way that it changes should make sense. If it doesn't agree with your intuition that doesn't necessarily mean that it's wrong, but if you do get surprises like that, you should be able to reason through and figure out what's going on. If it really doesn't make sense at that point then you have to consider that maybe there is something wrong with your model or that there could be a bug or something." (Ryan I.)*

Being able to logically explain any behaviour helps Ryan I. to determine if the model is sensible. Note that this theme is raised here in conjunction with the theme of intuition, rather than data. This assumes then that the model being used must be well-understood, so that rather than comparing with actual data, your intuition about the range of values is what is referred to.

Virgil I.'s definition of the model making sense is that it is robust, *i.e.* able to handle any given input:

> *"We created the original function, we solved the end function, so we could just compare them. But I think that is kind of like a logical fallacy in a way. I didn't*

> *like it, got in arguments all the time because, first, it just felt wrong. I think we needed more. And I just need a feeling that I've made it robust. And that word kind of captures my feeling, and what would qualify as robust to me is: it almost exceeds expectation in a way." (Virgil I.)*

Virgil I. explains that it "felt wrong", echoing Ryan I.'s statement about your intuition guiding you. Virgil I.'s point about creating data for his model being a logical fallacy echoes Brian Wetton's response to this question and Ted's response to the following one (see Sections 4.8 and 4.9). One has to be careful not to use data to create the model and then use that same data to test it.

Tess I. does not speak about checking with the data *per se*. However by consulting with the experts in the field she assumes that they have some knowledge of how the data should be behaving:

> *"So you then go back to the people if you're working in the applied field and say, 'Well doing this, what does that physically seem to do? Does that seem reasonable?' If there are already no kind of models out there existing that give you some benchmark solutions then you always go back to ask yourself, 'Does that physically make sense?' " (Tess I.)*

She uses the same term "reasonable" but explains that this relates to the experts in the field's knowledge of the physical properties. On the other hand Reuben I. says that for him it is difficult to look at his mathematical solution and be able to see if it is physically making sense. He therefore checks to see if it makes sense mathematically.

> *"If I have a set of equations that I'm solving I make sure that, [for example] the travelling wave solutions will propagate correctly." (Reuben I.)*

While Reuben I. checks a known solution, like the direction of the travelling wave in his model, Livingston I. does a more rigorous check of existence and uniqueness:

> *"Yes of course we need to [...] so when we get the equation then from the more pure, rigorous set of mathematics you should (probably you don't want to prove it but) be aware that the equation should like be well-posed, it has a solution, it's unique and such things." (Livingston I.)*

All of these intermediates have different ways to check if the model or the solutions make sense. Whether they are using their intuition, the intuition of the experts, checking the robustness of the model or checking the mathematics, they all identify ways that tell them that the model and hence the solution, makes sense.

However, the experts explain that intuition for the problem may be wrong. Just because a model is mathematically correct this does not mean it is solving the problem it is supposed to be. Note here that while these methods are not as efficient as checking the solution against new data from the physical system itself, they are useful if the data is not available, which may be the case for several of the intermediates.

**Intuition**

The theme of intuition was only raised by one of the experts interviewed. However, it appeared four times when interviewing the intermediates. Reuben I. who comes from a physics background laments his lack of physical intuition:

> *"Yeah I mean for somebody who's done a degree in physics I actually have a surprisingly, like I'm not the greatest at physical intuition just like understanding like is this doing a reasonable thing?" (Reuben I.)*

The above statement gives the impression that for Reuben I. physical intuition would be a useful asset in determining if the solution determined is a viable one. This is similar to Danny I.'s discussion about the use of intuition in determining what makes sense:

> *"We did some simulations so, the simulations kind of agreed with the intuition of what should be happening. And yeah that's what I would do, just try out a couple things where I know what the answer is and see if my method catches that." (Danny I.)*

Danny I. couples her intuition about the problem with use of test cases, where she already knows the answer, to determine if the solution makes sense. Ryan I. also uses intuition in understanding what your solution should be doing. Ryan I. also acknowledges that intuition may not be reliable, as the problem or solution may be counter-intuitive.

Intuition is defined as "a non-sequential information-processing mode, which comprises both cognitive and affective elements and results in direct knowing without any use of conscious reasoning" [53]. It is interesting that the theme of intuition appears more frequently than with the experts. A reason for this may be because the intermediates see the experts at work, but not knowing all of the steps taken, may attribute the experts' skill to their unconscious understanding of the problem. However, the expert does not see it this way. They understand all of the steps they take and know what experiences they are drawing from. This experience allows them to go through the process very rapidly, however, and may seem unconscious to those who are not fully aware of what they are doing. One of the principal ways that experts process so rapidly is that they aggregate knowledge into schema or organised conceptual structures, that guide how problems are represented and understood ( [54], p.33). In other words, experts are able to combine isolated facts that relate to each other, thinking of information in chunks as opposed to isolated facts. This organisation of information into chunks or schema allows the experts to process more rapidly than the non-expert who is sifting through the individual facts.

### Convergence Tests

Convergence testing is used with numerical methods, to check the numerical scheme. The theme of checking that the solution is sensible by convergence testing is one that was broached by two of the intermediates but not raised by any of the experts. Both intermediates that spoke about the use of convergence testing have strong backgrounds in numeral analysis, and partial differential equations:

> *"Yeah. I mean, in terms of PDEs you often have convergence studies, so I do those at every step. So as I said, when I started my modelling, I didn't start with the biggest PDE, the full PDE that I wanted, I started with the heat equation on a sphere and then worked my way up to what I'm doing now. But every step of the way you do convergence studies on solutions that you know hopefully. And if you don't know them you just refine your grid and hope that you're getting closer and closer to a solution." (Frank I.)*

The procedure of convergence testing is well known in the study of PDEs, and particularly the numerical solution of them. Why then have no experts raised this as a viable method for checking if the solution is sensible? The two intermediates acknowledge that they hope to converge to the correct solution. The experts also know that convergence studies while important to check that the model is working, does not give us enough information about the solution. This is simply a part of modelling for the expert, while for the intermediate, this is a step that holds some significance. This does not mean that the experts do not do convergence studies, but those studies are not what they use to check to see if the solution is sensible.

## 5.9 Changing Strategies

**What makes you change solution strategies?**

This question elicited two main themes among the experts, of being stuck and getting unsatisfactory results. These themes also emerged among the intermediates, although their language for describing them was not as concise as those of the experts. Three new themes were identified by the intermediates, changing strategies because: of input from others, usually more expert than themselves; of a desire to improve the robustness or efficiency of the numerical scheme; or to verify that the first method was correct.

**Being Stuck or Getting Unsatisfactory Results**

Four intermediates explain that if you are not getting a sensible solution, then you should change strategies. Ryan I. couples the theme of being stuck with the idea of understanding why one is stuck:

> *"At some level you have an intuition if you get stuck somewhere: am I stuck because I've done something fundamentally wrong and I need to go back and change my model? Or am I stuck just because there's something a little less serious I just don't understand this one thing [...] I guess that intuition is just coming from experience." (Ryan I.)*

Ryan I. also brings up the theme of intuition or innate understanding, which is found throughout his answers. Ryan I. accredits this ability to understand why you are stuck to experience. Liv I. approaches the same theme slightly differently. She does not clarify here whether she means that she is completely stuck, but also thinks it important to understand why one is stuck:

> *"You try a way and if it's not seeming to work very well then trying something else. For me [that] came when I read some of the research you see people are doing in a couple different ways. So you pick one, see how it works for your*

> *problem, and then you can change if it's not working for your problem. But I think there's learning in that too if you can understand why it didn't work using that method." (Liv I.)*

Liv I. uses the literature to determine what method to try, but does not advocate using the literature blindly. It is important to understand why a method is chosen in the literature and whether it is appropriate for your particular problem. Tess I. makes the point that unexpected solutions do not necessarily mean that you are incorrect, but can mean that you have broached new territory in your area:

> *"If you try something and you're clear that it's not worked, you have to understand why that's not worked [...] You have to say: why is it not giving me that solution? They could all be wrong and what you're doing is correct so you have to try to understand why what you've done is giving you the solution that it's giving you [...] it's by looking at the result and saying well is that physically reasonable or can I explain it in terms of the approach I've taken." (Tess I.)*

This idea that everyone else could be wrong is an important point, because if you are changing your method because you are not getting the expected solution, you may never advance the field with new ideas or methods. It is not surprising that as a PDF she seems to be the only one thinking this way.

But of course we can also get results that simply do not make physical sense and cannot be explained. This is another indicator that we should change our approach to the problem:

> *"You start in one direction, the one that you think is the most promising. But then for example if you decided to use forward Euler to solve this PDE, and then you see that you're getting gibberish, just junk, when you compare it to what you think you should get [...] Then maybe you try another method and see if that will give you better results. And then obviously if the other methods don't give you results you have to go back and see ok, well is the PDE itself even solvable, or well-posed? I mean this is me because I'm a little impatient, but ideally once you have the PDE the first thing you want to do is show uniqueness and existence and solvability and all those things." (Frank I.)*

Frank I. touches on his own impatience and desire to get to a solution before checking existence and uniqueness of the solution as he should. This is in contrast with what he thinks he should do, but the fact that he knows this is incorrect indicates he has passed the novice stage [51].

Ryan I. explains that if the results you get are sensible, maybe they do not answer the question you are trying to investigate, and hence a change is required.

> *"It depends on whether the way you have it right now is able to do everything or investigate everything you want to investigate. It could be that you come up with a model that answers some of the earlier questions that you have, but then you want to dig a little deeper and then if your model's no longer sufficient then you might have to go back and start off using something a little more sophisticated." (Ryan I.)*

Finally, Reuben I. explains that he does not worry about understanding why his problem is not working, but is more interested in getting the model to work. This is very different from the other intermediates that speak about the importance of understanding why you are stuck.

### Other People's Input

For two intermediates, collaboration with experts is important, deferring to the expertise of those that supervise them and have done it before:

> *"I think a lot of it would be other people's input. We thought we were getting good results for our poster presentation and then [the professor] pointed out that the equation we were modelling we didn't break it down properly into its components [...] so collaboration again will always help you. Make sure you check with someone who has a little bit more information than you do on that subject maybe. That will always help." (Isabel I.)*

This response illustrates how modelling with others especially those who are more expert in the area, can help. The intermediate is also not passive, as she goes on to explain that she fixed the problem even though the professor pointed out the error. We will see that this is in contrast with the complete novices who exhibit learned helplessness (Section 7.6).

**Improve Efficiency**

Changing strategies usually comes because there is something wrong with the original method, whether it gives no results, bad results or is just too complicated. However there are other reasons for change even if the results are correct. Linus I. explains that the efficiency of the model can also be improved:

> *"Normally when you're trying to find analytical solutions I'm happy if I can find one way. However I've been interested in finding multiple ways of doing the numerics for that problem. Because a lot of [times] there's better ways of doing it [...] if the program takes less time to compute the solution then that would be considered better. Also if you get a more accurate solution that would also be better." (Linus I.)*

Linus I. explains that a more efficient model is not necessarily easier to write or code, but gives a more accurate answer or is faster. This is similar to Virgil I.'s desire to improve the robustness of the model. Robustness is another quality that makes the numerical model more efficient as different code is not necessary for different inputs:

> *"It's like lightning striking it's going take a lot of little paths, but you don't want to be too close to the ground and then realise you have to go all the way back to the top and try a different path. You just want to branch out a little bit and go, 'Oh you know what? We should have written this part of the code differently.' Instead of, 'Oh we've got to change the basic data structure we're using.' So, but if you have to do that I would say, the main reason is because I couldn't make it a robust solution." (Virgil I.)*

Virgil I. suggests making changes as early as possible as opposed to finding out at the last minute that the entire method needs revamping. This is the reverse of Frank I. saying that he goes to the end of the problem and then works back to find the problem. There is some similarity here, since both want to avoid having to go through the entire process again. This is something that the experts accept may have to be done: "You should be prepared to change the model go back to the drawing board and start all over again." (Ted)

### Complicatedness of the Problem

Expert Sam Howison explains that if "the problem you're formulating is horribly difficult" then he would change strategies. Danny I. echoes this sentiment when describing her work:

> *"Let me think [10 second pause] well I guess in my thesis I did change some stuff in the middle because I had one model that was doing everything at once and it was extremely difficult to choose parameters, in a way that made it really undesirable and non-generalisable. So I decided to split it in 2 parts. Do one stage first where it's very simple and you can choose your parameter. And then another stage which takes over after the first one. I didn't see anybody do that. For my particular problem it made a lot of sense to do that but I didn't see it anywhere else before. So you know simpler is better." (Danny I.)*

The theme of simplification is a recurrent one throughout this work and here we see it again. If the problem is too complicated then instead of attacking it head on, stopping to simplify it makes more sense.

## 5.10 Difficult vs Easy

### What makes a problem difficult/ easy?

My expectation upon asking this question was that the experts would speak of the difficulty with the problem itself while the novice would have issues with the equations or mathematics. The theme that dominates the intermediate responses however is their familiarity with the problem, and whether they have solved it before. Other themes mentioned were: problem complexity, problem clarity and collaboration.

### Problem familiarity

For the intermediates, a familiar problem, one that they've seen and solved before, is by definition easy. This is why Frank I. describes textbook problems as being easy:

> *"In that case it's no longer a question of solving a new problem it's just a question of: 'Ok to solve all these new problems you're going to need to know how to solve this' [...] I think the main thing that that helps you with is transferring the word*

> *problem to a mathematical framework. In that case it's a simple sort of algebraic sort of thing. You'll have $ax = b$ and you'll have to solve that for $x$. It's pretty easy but I mean that's the base of everything right? So even when I'm solving my PDEs I'm solving an $Ax = b$, it's just my $A$ is a matrix, my $x$ is a super long vector and my $b$ is a right hand side vector. Essentially that's what you're always doing." (Frank I.)*

Frank I. believes that the textbook questions are easy because they are not new. He explains that this ease comes from the fact that the solution method is therefore known, and so to solve the problems, you only need to apply one particular known technique. He recommends transferring those base ideas to the more difficult concepts in order to get a handle on them. Isabel I. also believes that homework problems are easy ones, but because of their structure:

> *"An easy problem would be one where it was very structured, it would be like a homework problem, where they've laid it out to work out nicely. Whereas a hard problem would be one that maybe no one's looked at before, and you don't really know what to expect out of it. 'Cause homework problems you know 95% of the time they're going to work out, because someone's thought about them enough and used them enough that you know what your solution should look like. Whereas if you're venturing on new ground then it's not inherently a harder problem, but it's a lot harder to know that you're on the right track. There can be hard homework problems too but I think just knowing that there's a solution out there makes it feel easier in the long run, a solution that you can check." (Isabel I.)*

As Frank I. did, she touches on the fact that the homework problems are not new so you know better if you are on the right track. Having enough experience allows you to transfer that knowledge to unfamiliar problems. The unfamiliar problem then becomes easier to solve because it is seen as similar to a problem solved before:

> *"Simple also would be something that is very similar to an existing model, that has a widely accepted and reliable solution and you just change a few things to make it work for you." (Danny I.)*

Here the problem is not just similar to one that the modeller has seen before but it is also similar to existing well-known models. Having a known, standard, reliable solution makes the problem like those in the textbook, as you can check your answer to make sure that you are on the right track.

This familiarity with the problem and its solution is echoed by Liv I. Being familiar with the problem means that you have a better understanding and helps you to identify if you are on the right track with the solution. The familiarity of the problem also means that the solution is more tractable, even if the solution method is not previously known. Liv I. also mentions the ability to visualize the problem, and this sentiment is reinforced by Ryan I.'s comment below:

> *"I think it's how easy it is to visualize and how much intuition you have about the problem. A problem that people have seen in their daily life and have some intuition about, people are going to get a lot farther on it even if they don't have a lot of background in it. At least you have some understanding of what is the end goal and what you're really trying to model. Whereas if you're dealing with something where it's more abstract, just off the top of my head something like particle physics or quantum physics or something like that, I would have much more difficulty coming up with a model for that." (Ryan I.)*

Ryan I. attributes the ability to solve the familiar problem not only to the ability to visualize but to your intuition with regards to the real life situation as well.

### Problem Complexity

The complexity of the problem is something that makes it difficult but might not be obvious to the novice as being the real difficulty. A person bogged down with long calculations may not appreciate that the problem is actually easy because it is clear (though tedious) how to solve it. Some intermediates show an understanding of this as four of the 14 raise the theme of problem complexity when discussing what makes a problem difficult or easy.

Frank I. first speaks on being able to visualize the problem as his colleagues Liv I. and Ryan I. do, and raises the theme of intuition. He then discusses how the complexity of the problem structure itself can cause the problem to be difficult:

> *"Let's say for example right now you told me go code up something that models molecules bumping into each other. I'm going to say, 'Oh well that's super easy,' I just make a bunch of molecules, which are easy to implement. And then I make*

> *an interaction between them, and then run it and solve it. But the problem is there's things that I'm not going to think about. For example, if I have a thousand of them or a million of these balls, I need to think about computing power and stuff like that ( that's where neighbourless comes in, you're not going to compute the interaction between everyone, you're gonna compute the interaction between you and your 20 neighbours). My point is, if you think the problem's easy chances are it's going to surprise you, in terms of complications. If only as I said for numerical reasons." (Frank I.)*

This account shows by example how the structure of the problem can be complicated and thus make it difficult to model. This can be due to limitations on computing power, or simply the ability to keep track of what is going on. Danny I. captures this thought concisely.

> *"Let's see. Well difficult would be if there's too many things going on [laughs]. If the problem itself in nature involves too many things that change so you have to model a lot of things at the same time, chances are it's not going to be accurate when you put all the pieces together. Simple well, simple would be the opposite of that." (Danny I.)*

The simpler the structure of the problem the easier it is. Sam Howison explains that this is why some problems are notoriously complicated like atmospheric modelling. Two of the intermediates agreed with this comment, acknowledging that atmospheric modelling can be complicated due to the many influences that affect the model. Their solution is to find those influences that are most important or have the greatest effects and ignore the rest. Tess I. transfers this idea and explains that it can be done for all problems. This brings us back to question 4 (Section 5.4) and the importance of determining what terms should be discarded from the model, in order to acquire a model that is solvable with the given resources.

**Problem Clarity**

Problem clarity relates not to the nature of the problem itself, but to the ability of the experts in the field, those who want the problem solved, to articulate exactly what they want. Five experts including Sean Bohun and Lou Rossi raise this theme, as seen in the previous chapter. Lack of problem clarity is also mentioned by intermediate Virgil I. as something that makes problems difficult:

> *"To me it's just if they poorly define what they want. If someone says, 'I want to model a car.' What? Does it have to run the engine in my model? Or go straight? Does it have to be able to collide with someone? And I find that the biggest difference when I get a question from someone who really understands what they're doing versus doesn't is how well defined they even make the question. I find this is both math understanding and a communication problem. Just not knowing what they want. There's lots of reasons you could get a poorly defined question. And that could be poorly defining what they want to model or what they want to get out of the model." (Virgil I.)*

Having a really open question makes the problem more difficult to model, as we need to know what exactly we are modelling and why in order to do it effectively.

#### Poor Collaboration Skills

The theme of collaboration has been raised several times throughout the interviews with experts and intermediates alike. What is interesting is that no one except Danny I. below, has mentioned it in relation to making a problem easy or difficult:

> *"Difficult, would be working in a large group of people that don't know what they're doing [laughs], and that don't have that patience we talked about earlier? I think working with a lot of people is the biggest problem. Depending on the setting of those people [it] could be really, really great, or it could be a disaster if you have to explain everything a hundred times over." (Danny I.)*

The fact that collaboration is deemed a useful tool by so many of the experts and intermediates only helps to make this point more valid. Poor collaboration can make the mathematical modelling process a nightmare. We will look at the implications of this very important theme in the final chapter of this work (Chapter 8).

## 5.11 Summary

This look at the group of intermediates showcases several differences and similarities between the intermediates and the experts. The intermediates tend to provide more detailed responses on many of the topics, whereas experts were more succinct. The dichotomy in the definition of modelling is not evident as with the experts, with intermediates defining modelling as the use of mathematics to solve a real world problem. There is a shift in focus here from formulation or the entire modelling process to the solution step of modelling.

Intermediates are more forthcoming with their feelings on modelling than experts and have more feelings of persistent self-doubt. Intermediates also discussed trying to understand the problem initially, however they make use of research primarily to do this as opposed to collaborating with others. Intermediates recognised that the complexity of the problem often leads to it being difficult, as well as a lack of clarity and the openness of the problem.

Several of the themes mentioned by the experts re-emerged here. While intermediates have less autonomy, they are still usually interested and motivated in their particular area of study. They recommend asking questions when stuck highlighting that taking a step back or articulating your difficulties often helps you to overcome them. They named several mathematical areas of knowledge that were seen in the expert responses and recognised breadth as opposed to depth of knowledge as being important. Intermediates valued non-cognitive skills of perseverance, good collaboration and taking a break.

# Chapter 6

# Novices

Apart from the interviews of SFU graduate students and post-doctoral fellows, I also gathered data from undergraduate students from the modelling course Math 461. I obtained questionnaire responses from the students of this class who also participated in the mathematical contest in modelling (MCM). Competitors in the MCM choose their team members (a maximum of three persons per team) before seeing the problems. There are two possible problems given to work on and team members cannot seek help from anyone outside of their group. The teams work on the problems for four days and are then required to submit their solutions (which may be partial solutions). These undergraduates that participated in the MCM were on the novice end of the spectrum, as this was their first modelling class and their first modelling contest.

Questionnaires were filled out by most of the students a few months after the competition. The questionnaire had been given immediately after the MCM; however many of them had several other commitments, and while agreeing to fill it out, they required some reminding and prompting to get the results in. Because of this in some cases they could not remember exactly what they had done.

Below is a discussion of their responses to each question. Note that these eight students are anonymous and the initial 'N.' after a name indicates novice status.

## 6.1 Question 1: Defining Mathematical Modelling

**In your opinion, what made this problem a modelling problem?**

This question parallels the first interview question on defining mathematical modelling. While I did not believe that these novices would be able to give a complete definition of mathematical modelling, I expected that they would have some framework in their minds which helps them to identify modelling problems.

Similar to the intermediates interviewed these students do not describe modelling as the experts did. Instead, they speak of modelling simply in terms of using mathematics to solve real world problems. Thus as with those interviewed the most common themes to emerge are: real-world problems and the use of mathematics, as seen by the three below quotes:

> *"It asked for a mathematical representation to a real life problem." (Phil N.)*

> *"Setting up a physics-based problem." (Mark N.)*

> *"This was a modeling problem because it involved using math to represent the problem and simplifying it with assumptions." (Jamie N.)*

Mark N. is even more succinct than Phil N. in his description, forcing us to assume that the physics based problem is a physical real life one, and it was set up using mathematics. Jamie N. goes a little further raising the themes of assumptions and simplification that have been identified in the two previous chapters. Others provide additional depth by giving some details about their problem:

> *"In my opinion, this is a problem of designing a half-pipe in order to fully explore the players' skating skills, it has its practical meaning. And there are many variables that we have to control and those variables are affected by each other, in that sense, this problem falls in the modelling category." (Karl N.)*

Karl N. gives a definition by example, explaining how their particular problem was a real life problem by defining it as "practical." He also comments that there were many variables which contributed to this being a modelling problem. From the responses below we notice that the themes of an optimal solution, and having more than one solution method or approach were also mentioned:

> *"There exists an optimal value to be found and under different assumptions, this value might be distinct but similar. The essential thing is the problem can be solved by many different ways ... no exactly correct answer." (Harry N.)*

> *"I believe the openness of this problem allowed many possible ways to model this problem mathematically. For instance, cross-sections of the half pipe could be modeled by using parabolas; relevant information could be found by examining arc length (these would reflect the distance a snowboarder would have to travel) or curvature (reflecting steepness). There were many other ideas that were brought up by group members too, and most of these ideas were unrelated to one another, implying that this problem had many different approaches to it. Thus I believe the flexibility of this problem made it a modelling problem." (Joe N.)*

> *"This problem is a modelling problem because we had to go through lots of assumptions to simplify the problem and reach a logical solution. Also, there was no right or wrong answer especially since not enough information was given. We had to go through some research, but due to the time, we couldn't possibly have a clear image of what the question was asking." (Karen N.)*

These students appear to be defining a modelling problem as any problem that is open, flexible and possibly ambiguous. In contrast, the experts identified these qualities as the side effects of working with real life problems that have many variables and are therefore impossible to model exactly. This is especially interesting as these three students were in the same modelling class. It is quite possible that because the fact that modelling problems often have no exact solution was highlighted in class, these students believed that this is what defined a modelling problem.

## 6.2 Question 2: Initial Thoughts

**a) When you first saw this problem, did you think you could solve it?**
**b)Why / Why not?**
**c) How long did you think solving the problem would take you?**
**d)What aspects of this problem were familiar?**

This question was designed to identify the first thoughts upon seeing a problem. Did the novices questioned feel confident or unsure upon first viewing their problem? Five of the responses showed that at least at first these students were confident that they could solve the problem. One novice was more neutral in his/her answer, while the final two did not feel confident they would be able to solve the problem.

Of those who felt sure they could solve the problem, two of them thought that the problem could be solved within a short period of time as well:

> *"a) Yes.*
> *b) It seems like a physics problem and I'm a physicist.*
> *c) 3-5 hrs.*
> *d) Mechanics." (Phil N.)*

> *"a) Yes, I thought I could.*
> *b) We chose problem a...it appears like a high school physics problem.*
> *c) Then I thought ... this must be easy... it could not take me more than 1 day*
> *d) Energy conservation law, newton's law, friction." (Harry N.)*

The problem that these two students each worked on in the MCM was a physics-related problem. The perceived ease of the problem was based on the self-proclaimed expertise of the first student and the fact that the second student viewed the problem as a high school problem.

Three other MCM competitors also viewed their problem as solvable, but expected it would take the entire weekend allotted to solve it:

> *"a) When I first saw this problem, I thought I was going to be able to solve it.*
> *b) The answer to the problem was based on common sense.*
> *c) A few full days.*
> *d) Setting up the physics and solving the PDEs." (Mark N.)*

> *"When I first saw the problem I thought was that was solvable because it seemed to be quite intuitive. I thought that it would take all the allotted time in order to model it well. This problem required velocity, distance, displacement calculations that were very closely related to the material in physics class." (Jamie N.)*

The reasons that Mark N. and Jamie N. give for their perceived ease of the problem is the fact that it was intuitive or "based on common sense." The physics of the problem also seemed to be familiar to them. Juliet N.'s response is slightly different, as she notes that her initial assessment of the problem was not accurate.

> *"a) Yes.*
> *b) Because at first glance it seemed easy, after starting however, I realized how complex it was going to be / how many simplifications were needed.*
> *c) The whole weekend.*
> *d) The physics of snowboarding." (Juliet N.)*

Juliet N. discovered that the problem had a hidden depth as was discussed by Bob (Section 4.10). What she believed to be easy turned out to have complexity she had not anticipated.

Karen N. does not state whether or not she thought she would be able to solve the problem at first glance. For her, the first stage was to try to understand the problem. This is in direct correlation with Polya's first step in *How to Solve It*:

> *"a) Some people in my group thought they had an idea of what type of question it was, so the first stage was to understand the question and the direction of how to solve it.*
> *b) We couldn't ask the professor, and we didn't have enough time to do enough research.*
> *c) It was hard to tell, especially that we didn't have any previous knowledge on the subject.*
> *d) We thought it had a relationship to some physical aspects, like forces at certain angles, and that's where we started." (Karen N.)*

From Karen N.'s response we can infer that she was not completely sure that she would be able to solve it, or if so how long it would take her to complete it. Her comments that there was not enough time to do research and that she could not contact a mentor, suggest that for her these are favoured heuristics. This is in keeping with nine intermediates interviewed including Liv I., Isabel I. and Virgil I. among others (see Section 5.3).

Two other MCM participants explain that they did not believe they would be able to solve the given problem. For Karl N., this was due to ambiguity in the problem statement and a need for more time:

> *"a) No, absolutely clueless, so did my teammates.*
> *b) Because there exists ambiguity of the way the problem is presented, maybe it is*

> *our problem that we cannot understand the problem well, plus everything seems to be so implicit, that we have to think of everything.*
> *c) Maybe a week would be proper for us, including writing the report*
> *d) Physics." (Karl N.)*

Karl N. appeared to be overwhelmed claiming he had "to think of everything," which is similar to Joe N.'s concern about the problem. He explains that he had to "start from scratch" below:

> *" a) No.*
> *b) Most of the math problems I have seen modeled in the past had to do with observing trends, rather than to do with designing something physically. Furthermore, for this problem, we had to start from scratch - most [of] the problems I have seen before had someone guiding us through.*
> *c) I can't quite answer this question since "solving this problem" doesn't really mean much in this case. Creating a half pipe isn't too difficult, but maximizing height isn't that easy. There are always tweaks to increase the height by more.*
> *d) Maximizing height was the most familiar aspect. I've seen problems in the past where we had to maximize or minimize certain things." (Joe N.)*

The absence of understanding, familiarity and guidance all contributed to their lack of confidence in their ability to solve the problem even after the fact. These are all themes that have appeared in the two previous chapters. The novices illustrate how not having these aspects when they are deemed necessary can affect the students' confidence in their ability to do the problem.

## 6.3 Question 3: Initial Barriers

**a) Did you start right away?**
**b) Why? Why not?**

The responses to this question were split, with five of the participants saying they did start immediately and the other three saying that they did not. Of those who did not start right away, the main reason was to allow some time to understand the problem. MCM contestants Karl N. and Karen N. explain below:

> *"a) No, we thought about it for a bit, and did some research online that afternoon, maybe we had a sound rest that night.*
> *b) Because we have to fully understand what this question wants us to do, and have to decide what approach we are going to use before we start." (Karl N.)*

> *"We couldn't start right away because not everyone in the group was available, and more importantly we needed time to figure what the question was asking. We also had to go through some research to have a better idea on the subject." (Karen N.)*

This desire to get comfortable, research and understand before attempting it is in line with Polya's first step: understand the problem [34]. Students often tend to rush into trying to understand the problem without fully understanding exactly what they are supposed to be doing.

Other reasons for not starting immediately include not being there and being overconfident that the problem could be solved.

> *"I joined the team afterward so I wasn't able to start on it right away. By the time I joined my two other team members have already worked on it a bit and so I was able to take a look at what they already have and expand on it." (Jamie N.)*

> *"No... since I thought it was easy, I relaxed..." (Harry N.)*

One student claims to have postponed starting for one day due to time constraints:

> *"Due to time constraints, I started the next day." (Mark N.)*

This is notable because of those who said yes they started immediately, the most common reason for doing so was the constraint of time.

> *"a)Yes.*
> *b) We only had about 3 or 4 days, so time was quite valuable." (Joe N.)*

> *"a) As soon as I got in contact with the rest of my team members I did.*
> *b) Because we had limited time to meet." (Juliet N.)*

For these students, the limited time appears to have overridden the need to understand the problem first. We also must take into account that for some students, trying to understand the problem may be seen as starting.

Finally, Phil N. did not give such a clear reason for why he started immediately:

> *"a) Yes.*
> *b) Why not." (Phil N.)*

This type of response is significant in its lack of details. There are several reasons why starting immediately might not be ideal, such as a lack of understanding, not having a clear plan. By responding "why not," it implies that Phil N. is not aware of reasons for waiting to start solving the problem, or is unwilling to answer the question.

## 6.4 Question 4: Determining what is relevant to the model

**What information did you decide was unimportant? Explain.**

Determining what information is relevant to the model is very important in the modelling process; the experts and intermediates interviewed also agree on the difficulty and importance of this step. The theme of simplifying and always using the simplest model first re-occured time and again in the previous two chapters.

Interestingly for two of the MCM participants, there is no recollection of identifying information that was not important:

> *"All the information I recall was important." (Joe N.)*

> *"I didn't find any unimportant information." (Jamie N.)*

This is even more interesting since their colleagues who worked with them did recall ignoring certain aspects of the problem in order to make it more tractable. In complete contrast, MCM contestant Karl N. thought that there was a lot of information that needed to be discarded, leading to constant discussion:

> *"Sorry there are too many of them, we are discussing all the time." (Karl N.)*

However, Karl N. did not give any details about what he determined was necessary to ignore in order to develop a working model in the time frame allotted. Five other novices were more descriptive. These details fell into two categories: assumptions added or simplifications made. Mark N., Phil N. and Karen N. made assumptions about the problem that were not specified:

> *"I assumed there was a symmetry, so I ignored the entire case." (Mark N.)*

> *"A person's speed, for the course should be optimized for all persons or assume the person most skilful." (Phil N.)*

> *"For part A, we actually had very [little] information given, so we had to make our own assumptions, and decide what shape might the tube look like. We also started thinking why it wouldn't look like the standard shape that is used by skaters." (Karen N.)*

Karen N. explains this is because the problem was particularly open and enough details were not given. This forced the group to make assumptions as deemed necessary. Two other MCM contestants Harry N. and Juliet N. explain features that were removed to simplify the model as opposed to assumptions that were made:

> *"I think how the athlete pump was unimportant...actually... I could not model it....so I simplified the model in another way that assume the athlete did not pump but an high initial velocity was given before he entered into the transition part... the high velocity is given that could guarantee he could fly out of the edge..." (Harry N.)*

> *"The friction of snow and the pumping technique because we wanted to simplify the problem by having gravity and the normal force as the only two forces acting on the snowboarder." (Juliet N.)*

Juliet N. explains that she wanted only two forces but did not explain why the other forces were unimportant. Harry N.'s reason for his simplification is not based on the effects that the pump had on the problem as much as his inability to model it. This is at odds with the responses of the experts and intermediates interviewed, who explain that removing the variables that are having the least effect on the overall dynamic of the problem is critical for simplification (Section 4.4).

## 6.5 Question 5: Mathematics Used in Modelling

**a) What mathematics did you use or try?**

**b) Was there anything you thought would be useful that was not?**

The reason for asking part a) of this question was to discover what mathematical content knowledge was needed to participate in the MCM. Students used a broad and varied skill-set of mathematics including PDE solutions, ODE solutions, numerics, calculus, Euler-Lagrange equations, mechanics, physics and calculus, with partial differential equations being the most used here. This is in keeping with the mathematics that the experts recommend.

Part b) of this question stemmed from my own observations of work at mathematics modelling camps. Bob also talks of problems being difficult due to "hidden depth," and intermediate Danny I. explains that once they were working on a problem and someone discovered that the scale with which their problem was set had in fact made their method incorrect.

I was therefore interested in seeing whether this was something that these novices recognized for themselves as happening. Seven of the eight students answered negatively to this question. Four of them could not remember or simply said that this did not happen:

> *"a) PDE related information.*
> *b) None." (Mark N.)*

This may be simply a case of not remembering the situation, as Phil N., Juliet N. and Jamie N. explain in their responses.

> *"a) Euler-Lagrange equations and formulae of mechanics.*
> *b) Can't remember." (Phil N.)*

> *"a) Mainly geometry pertaining to circular objects and physics concepts.*
> *b) Can't remember."(Juliet N.)*

> *"We tried to solve it from physics aspect, using calculus. We thought that we could calculate the total force on the snowboarder and then derive acceleration then velocity from that. Then we could plot the trajectories of the jumps using parabolas. I'm sorry but I don't remember if there was anything that we thought might be important but was not." (Jamie N.)*

This lack of recall is possibly a result of the time lapse between partaking in the exercise and filling out the questionnaire. There is also a possibility they did not consider anything that they later discarded or that the parts that were unimportant were not significant enough for the participants to remember.

Two other MCM contestants explained that their reason for going with the initial plan only, was due to a limited time frame:

> *"a) We used so much physics and calculus, and ordinary differential equations.*
> *b) We hardly had enough time to think of a method and go with it, so we didn't have enough time to change our opinion." (Karen N.)*

> *"a) Lots of physics was involved (which I left to my partners to solve). I attempted using graphs (I guess today I would call it curve fitting, but back then I hadn't studied this yet) to model it. Furthermore, rates of changes were quite important, so basic calculus was key.*
> *b) Perhaps due to the time constraint, as soon as we thought of something that sounded like it worked, we just went with it." (Joe N.)*

This response to the pressure of time is a negative one, as it stops students from identifying whether a solution is viable or not. Karl N. below makes the interesting point that everything used was valuable in some way:

> *"a) We tried both continuous and discrete approach, since we found that by using continuous method, we ended up with a bunch of nasty differential equations that we could not solve by hand or software, then we switched to discrete method, but we present both approaches in our report.*
> *b) I cannot say anything we thought of was useless, they are all valuable in some ways, we put everything we thought of on our paper." (Karl N.)*

Although we cannot discount the possibility that the time lapse between the contest and the responses to the questionnaires may have caused novices to forget what aspects they discarded, these responses appear to reinforce what is seen in the literature: novices tend to go ahead with their initial plan no matter what [20]. In contrast, MCM contestant Harry N. shows some expert behaviour when he makes mention of his change of mind from the use of inertia in the problem to not using it:

*"a) Integral.....however, it could not be integrated by hand.... I mean the work done by friction... so, I used midpoint rule...to get more accurate, I used many points to ensure the accuracy.*
*b) I firstly thought to think about inertia...but finally did not think it that way."(Harry N.)*

This tendency of the novices to stick with one solution method reinforces the fact that although breadth is key, it requires the mathematical maturity to know when to use what and how the different areas are interlinked.

## 6.6 Question 6: Dealing with Being Stuck

**a) What barriers did you encounter?**
**b) How were they dealt with?**
**c) Did working in a group help or hinder your progress? Explain.**

The question of being stuck and what to do when one is stuck has been dealt with in the two previous chapters (see Sections 4.6, 5.6). From this questionnaire I wanted to gather a first hand report of when the participants were stuck and how they overcame it. Part c) of the question was added to gauge if the students, who did not have a lot of group experience, would favourably view the group dynamic. This decision was made after some analysis of the expert interviews where the theme of collaboration was seen as a significant factor in mathematical modelling.

In the previous chapter (Chapter 5), Danny I. discussed her problematic experience when working with others who do not have the required collaborative skills. At the intermediate stage of expertise, the benefits of collaborating with those more expert appear more valued than collaborating with those who appear to be less expert. This led me to wonder how the participants of the MCM found the collaborative aspect of the work especially due to their limited experience collaborating on modelling problems, and knowing no expert mentor was available to them. This is especially in light of how they overcame barriers, since collaboration is the main way used by the experts to do so.

The first response is not very helpful. We see that Mark N. was stuck when trying to justify his result and never resolved this issue:

*"a) Justifying my final result.*

*b) Left unjustified.*
*c) Not really." (Mark N.)*

His comment "not really" to part c) of the question does not tell us if he thinks the group aspect of the MCM was a help or a hindrance. Similarly Harry N.'s answer to part c) below, does not really answer the question:

*"a) How to calculate the work done by friction.*
*b) I used numerical way to a approximate the work...use matlab.*
*c) I firstly thought about it by myself... then communicated with them... they said okay.. go [a]head...see what we could get...one of our members thought mine method should not get an optimal answer due to the curve we cannot assume." (Harry N.)*

What is helpful here is that we see that he resolves his problem by using numerics. This is similar to Karl N.'s method for becoming unstuck. When numerics did not work for Karl N. he chose an alternate method:

*"a) I might be the one who had most barriers in the team, since I am not in the modelling class, and never learned ODEs and PDEs. the most unforgettable barrier that our team encountered was solving those nasty equations.*
*b) We tried matlab and maple to solve them, but we didn't get anything from them, so we just gave up doing that, and switched method.*
*c) No, I really learned lots of things from my teammates, and from this experience, that will for sure benefit my future study or next MCM." (Karl N.)*

Karl N. also commends his group members with helping him. This is echoed in Joe N.'s response, although Joe N. implies that the numerous ideas that came from working from a group were a possible issue:

*"a) i) Some of the definitions presented in the problem weren't quite clear.*
*ii) At times there were disagreements among group members.*
*iii) Some of the models we created lead to nowhere.*
*b) i) We had to make many assumptions.*
*ii) Nothing you can really do here.*

> *iii) We had to immediately discard the model and try another way rather than fixing it (again perhaps due to the time constraint).*
> *c) It definitely helped. I recall one group member was great with physics, while another member had more programming experience. Also, more members meant more viewpoints and ideas." (Joe N.)*

It is interesting that Joe N. speaks of models that "lead to nowhere" and were discarded, because in response to the previous question he claims to have used everything that was tried. This perhaps clarifies that for Joe N. anything that was used at some point was useful, even if it did not work and had to be discarded afterwards. Again, this might be attributable to the time lapse between participating in the MCM and answering the questionnaire, causing him not to remember what was discarded when trying to answer the previous question. Joe N.'s method for becoming "unstuck" was to try an alternate approach.

The remaining MCM students are ambivalent when it comes to deciding if working with a group was a help or a hindrance:

> *"Working in a team was helpful because instead of just me trying to figure things out, I was able to talk to other team members and we would fill in holes in each others' reasoning. There are times when it hindered the progress because we couldn't agree on how to best approach the problem, but overall it was better to work with team members." (Jamie N.)*

> *"Some of the barriers were dealing with working with students that weren't in Math 461. Also, not having enough background knowledge and having limited resources. At some stages, being in a group helped to counteract and arise a discussion about why an idea won't work, or sometimes to build on a logical idea." (Karen N.)*

> *"The physics part was a hinderance since my group members didn't know much physics, but the later numerical analysis I wasn't much help." (Phil N.)*

> *"Both; hindered because everyone had such a wide range of ideas and it was hard to decide on one approach, helped because each of us had our own set of strengths." (Juliet N.)*

Ways to become unstuck here were making more assumptions and possibly simplifying the problem, as well as using scientific computing (also called numerics) to solve analytically complicated problems. Interestingly, the other barrier that these students encountered was that of working in the group. This was caused by several factors: the other students did not have the same background knowledge, were unresponsive, or had contradictory ideas. It is because of this many of these students had issues with the un-mentored group dynamic, although they recognized there to be some value to it.

## 6.7 Question 7: Problem Solving Heuristics

**a) How many different problem solving strategies did you try?**

**b) What were they?**

**c) If you switched strategies, what caused you to do so?**

This question is an amalgamation of interview questions 7 and 8b). Part a) and b) mirror question 7 of the interviews on heuristics. Here I used the term "problem solving strategies" as opposed to heuristics. Part c) of this question is the same as interview question 8 b).

The experts provided several heuristics for mathematical modelling: drawing a picture, exploiting a simpler problem, exploiting a similar problem, collaboration and looking at limiting cases. The MCM students' responses suggest that they did not see the question as asking for problem solving techniques as opposed to specific mathematical techniques. However, we can still glean information from their answers as to why they switched strategies.

The theme of being stuck is raised by four of the MCM students as a reason for changing strategies:

> *"a) I recall I had a bunch of ideas at the beginning, but I didn't have a strong enough physics background to back them up so most [of] my big ideas were discarded. Eventually, I was just programming for the most part.*
> *b) To be honest, I don't really remember anymore.*
> *c) Switching strategy was due to a shortcoming of the first strategy. If we bumped into a big problem, we often didn't spend time trying to fix it."* (Joe N.)

> *"a) A lot; we re-started our thinking / design at least around 5 times.*
> *b) Researching, simplifying, etc.*

*c) Because the other strategy was resulting in problems." (Juliet N.)*

*"a) A few.*
*b) 2D, 3D, different surfaces.*
*c) Getting stuck." (Mark N.)*

*"We probably only tried two strategies. We first started by thinking of the actual shape, and why won't any other shape work. Then we figured we'll start with the standard shape that skaters use, and determine the angles that would produce the highest jumps. It was hard to determine what the question was asking, and why have people decided to build it the way it's built. We switched the strategy because we were getting stuck and running out of time, and we had to do something that would convince the judges, even though we were still not convinced by our own strategy by the end of the competition." (Karen N.)*

As with the intermediates that were interviewed, all of the responses did not explicitly use the term "stuck." Some students describe this as "the shortcomings of the first strategy" or "the other strategy resulting in problems." The theme of time pressure is also evident here in Joe N. and Karen N.'s responses. Karen N. also re-iterates a concern about the ambiguity of the problem given. This is not surprising, as we know one of the big difficulties of working with mathematical modelling for students is the openness of the problem. The experts deal with this by going back to the expert in the field for more details, but this option was not available to the MCM students.

A complicated problem was also a reason for expert Sam Howison to switch strategies. This ties in with the recurring theme of experts and intermediates alike to use the simplest method possible, and is echoed by three of the MCM contestants:

*"a) We tried a very complex physics way... and my numerical way.*
*b) The complex physics way is actually out of my knowledge.. one of our members came up with it... he is physics major.. but he cannot solve it finally... he asked me to programm a numerical way....HOWEVER, it was too complex, I could not even help it out although I thought my numerical skills was good.*
*c) After failing his method, I came up with my method because of failing to solve....I used a simpler model and numerical method..." (Harry N.)*

> *"a) Two.*
> *b) Continuous and discrete.*
> *c) As I explained before, since we can't solve those complicated differential equations." (Karl N.)*

> *"a) At least 3 I remember.*
> *b) Using the Newtonian mechanics, and Langrange mechanics, and numerical.*
> *c) Too complicated." (Phil N.)*

Of note here is that for Sam Howison the complexity was embedded in the problem itself, whereas the complexity of the mathematics or numerics was the issue for these novices. This illustrates the point made by expert Burt Tilley, "For students, lack of math knowledge hampers." This is reinforced by Bob's statement about what makes a problem difficult:

> *"Hidden depth. A problem may appear easy on the surface. For students: complicated equations for those who don't want to use computers, or analysis for those who only want to use computers, makes the problem difficult. This is why we should be working together." (Bob)*

One MCM student is the exception as he claims not to have changed strategies:

> *"We didn't exactly try to switch strategies. It was our first time modeling a real-life problem so none of us had much idea on what to do." (Jamie N.)*

## 6.8 Question 8: Experiencing AHA!

**Did you have a breakthrough or "AHA!" moment that helped you to make progress on this problem?**

For the most part experts prescribe collaboration as a means of getting unstuck. However, a few of them spoke of taking a walk or waiting for ideas to come to them (Section 4.6). My own work at modelling camps consisted of a few AHA! moments that were significant and had a big impact on my understanding of the problem. From Liljedahl's work [33] on this we know that what might be an AHA! moment for one person may not be for another. Thus I was interested in finding out if AHA!'s play an important role in modelling for the students participating in the MCM.

This question elicited a full range of answers. Students claimed to have AHA! moments that were helpful, AHA! moments that were not helpful, and no AHA! moments at all. Five of the MCM students spoke of having a helpful AHA! moment:

> *"Yes, that occurs in the last 50 min before it was due. We got an answer from the discrete approach." (Karl N.)*

> *"Yes there was such a breakthrough moment and we ended up using it for some parts of the problem." (Joe N.)*

> *"There was a moment when I realized it was simply an ODE." (Mark N.)*

What is interesting about these first three positive responses about AHA! moments is that even though the AHA!'s were helpful they were not elaborated upon. These first three descriptions of AHA! moments have almost no details in them. No information is given on how or why the students were stuck, what the big idea or breakthrough was, nor what caused it.

The other two students who had helpful AHA! moments, elaborated to a greater degree:

> *"Yes....I assumed the curve was a ellipse....and finally found out the solution....it was very close to the fact which is a semi-square....my optimal ellipse is close to a semi-square but not a perfect one." (Harry N.)*

> *"We had some "AHA!" moments while we were trying to figure out why one strategy or one assumption wouldn't work. We first tried to simplify it a lot like determining if it's 1D or 2D or 3D. We started by solving it 1D, then thought we need more variables and moved on to 2D." (Karen N.)*

Although more detailed in description than the first three, Harry N.'s description does not give much more insight. He articulates that he had an idea about the solution shape which he found to be erroneous, but it is not clear exactly what was the AHA! was for him. Karen N.'s description is similar in its vagueness. This may seem surprising because AHA! moments are moments of clarity, so we might expect that the description of those moments would also be clear. However, we know from Liljedahl's work [33] that students have difficulty describing their AHA!'s.

Two other responses suggest that although there were AHA! moments they were not particularly helpful in arriving at a solution:

> *"Sometimes, but then we'd just run into another problem and eventually ran out of time" (Juliet N.)*

> *"There were multiple "AHA!"moments, but they ended up not being very helpful in the big picture. We thought that finding out some of the information would help us with modeling the problem but they ended up not being able to contribute much." (Jamie N.)*

For Juliet N. the AHA!'s seem to lose significance due to the fact that immediately she was faced with other problems. Jamie N. says that the AHA!'s did not contribute much.

Phil N. is unique in his response as he claims no AHA! moments. He explains that everything went according to the plan:

> *"No, everything went as planned." (Phil N.)*

This is in contrast to his response to the previous question (Section 6.7) that he tried 3 different strategies, switching strategies due to complexity of the equations (Section 6.6). While his previous remarks do not indicate that everything necessarily went as planned initially, this response does suggest that plan was adapted as needed without a breakthrough or AHA! moment.

## 6.9 Question 9: Learning to Do modelling by Doing it?

**9. a) If you saw a similar problem, what would be the likelihood of you solving it?**
**b) If you saw another unrelated modelling problem, do you think you are better equipped to solve it?**

This question is aimed at identifying if the students, having been through this modelling experience, now feel better equipped to do more modelling. Experts agree that the way to learn how to do mathematical modelling is to do it. This ethos is similar to that of how to do problem solving. From Polya [34], to expert Sam Howison (see Chapter 4) the general consensus is to learn to do it via experience. However, Schoenfeld's [35] recurring theme of metacognition informs us that there must also be a big picture view of what one has done, and an effort to understand your own thinking process in order to improve it. This correlates to Mason's discussion that "experience alone is not enough [...] REFLECTING on the key ideas and key moments intensifies the critical moments of an investigation and helps to integrate their resolution into your thinking repertoire" [28].

The students' responses again went through the full range of answers, from complete confidence, to not being confident to say that a new problem would be solved. Phil N., and Mark N. both respond with confidence in their ability to solve any new problem, with very concise answers.

> *"a) Probably 100%.*
> *b) Ya, even if it is completely unrelated, since any exercise in prob[lem] solving helps solving problems." (Phil N.)*

> *"a) Probably I would be able to solve it.*
> *b) I would definitely be mentally prepared for it." (Mark N.)*

Phil N. echoes the premise that becoming a better problem solver comes from doing problems. Mark N.'s response is of interest because he does not credit the process with increasing his mathematical ability. Instead he explains that he would be "mentally prepared for it."

Harry N., gives more detailed response to the question, which is a little off topic but is still interesting.

> *"a) For sure I would solve it...and I think my solution to this problem is quite reasonable, but we do not have enough time to write the summary perfectly..*
> *b) Yes... modeling is really fun for sure..I want to participate again..however, I'm going to graduate after this term." (Harry N.)*

Harry N. gives the impression from his answer that he assumes that the modelling I am referring to is only in the MCM. Thus his impending graduation, which will make him ineligible for the MCM will stop him from doing any further modelling.

The other students interviewed give caveats on their ability to solve other problems. For Juliet N. and Karl N. the constraint of time would have to be lifted.

> *"a) Depends how much time I had*
> *b) Maybe; simplifying the problem would be the key, but the more it's simplified, the more unrealistic it becomes" (Juliet N.)*

> *"a) If giving us 2 weeks, maybe we can hand in a complete solution.*
> *b) Maybe, because physics is not my comfortable zone, I am better at stochastic process." (Karl N.)*

Juliet N. indicates an understanding that simplification and capturing the qualitative behaviour are important themes in modelling. Karl N.'s response implies that he would be better at solving a problem that he was more familiar with. This ties in with the theme of familiarity raised in the previous chapter (see Section 5.10).

Jamie N. Joe N. and Karen N. were all hesitant to say that they would definitely solve a similar problem.

> *"If I saw a similar problem I think I would have a better idea of how to go about modeling it. I don't think I will be able to solve it easily but then I would know how to start. If I saw a unrelated problem I think I would be better equipped to solve it. Because I have seen a mathematical modeling problem and worked on it with two other individuals, I have a better idea of how to go about solving it." (Jamie N.)*

> *"a) I think I would do better the next time I tried a problem like this, but I wouldn't be confident enough to flat out say I can solve it.*
> *b) Definitely. This helped a lot." (Joe N.)*

> *"We didn't actually try to solve it afterwards, or go into details to see where we went wrong, so I'll probably have an idea of how to solve such problems, but I still can't exactly solve it. However, Math 461 prepared us to solve a lot of different real life problems, but there's still a lot to come. Every math problem requires a different approach and different assumptions, but at least now I know several different approaches to solve a Modelling problem." (Karen N.)*

These students all seem to believe that the experience has better equipped them for solving unfamiliar problems. However, being better equipped does not necessarily mean that they believe they would be able to actually solve the problem. Jamie N. claims that she would have "a better idea of how to go about solving it". Karen N. clarifies that her ability to solve the problem is problem dependent. This was also mentioned by a few of the experts when asked how they deal with different aspects of mathematical modelling (see Chapter 4).

## 6.10 Summary

The novice modellers who participated in the MCM have a basic definition of mathematical modelling. However, some of them assume that modelling problems must have more than one solution method which, based on the expert responses, is not a necessary condition for a problem to be a modelling problem. The novices, like the intermediates, used research more than collaboration when stuck initially.

There was a wide range of emotions discussed upon seeing the modelling problems, with some novices feeling confident and others feeling completely clueless. There was much less autonomy than the experts on possible choices of the problems they were to work on. The novices did not always have a realistic idea of the time that would be required to work on the problem and did not have great collaboration skills, although they did recognise that collaboration could be useful. They did however indicate in their responses that they were willing to defer to others in the group and change strategies if they were stuck.

Many novices either stuck to one solution method regardless of the outcome, or switched completely with no effort made to understand why they were experiencing difficulty. They quoted several mathematical topics that they deemed necessary to solve the MCM problem they chose to work on. There was also no mention of checking that their solution was correct, but this may be due to the time constraint of the MCM.

# Chapter 7

# Complete Novices

To complete the spectrum of modelling ability in this research, I observed, worked with and acquired questionnaire responses from 45 students in 2 separate FAN X99 classes. The FAN X99 course is structured so that solving mathematics problems is the main focus. Early on in the course the students spend time working on basic number theory skills and their understanding of patterns. However after the first few weeks of this, the focus switches to translating from worded problems to equations and graphs.

These students often struggled with mathematics as a whole, and any type of word problem in particular. Here we are looking at the most basic type of modelling, where the problem is worded in a real world context, but is nowhere as complex as those problems encountered in industry. However the basic premise is the same: they are given a problem, which they have to translate into some mathematical framework and solve, in order to answer the question asked.

These students were given questionnaires at the end of their semester, with questions that paralleled those given to the experts and intermediates in many ways. Some of the questions however, were geared towards identifying any myths they might hold about mathematics and solving word problems. As before, some illustrative responses will be given to highlight the themes that emerge.

## 7.1 Question 1: Time needed to solve a word problem

**How much time do you think solving a word problem should take in general?**

This question was intended to identify how many of these complete novices had preconceived notions about the time required to solve problems. While the experts will speak of a year's work towards solving a problem, complete novices often believe all mathematics can be solved quickly or not at all. This is seen in the responses to the questionnaire. Students expect solving of problems to be on the order of minutes for the most part.

I sorted the responses in three main categories. The first category consisted of five-minute intervals or less, or exact answers. Fourteen students believed the time to solve a word problem was between 0 – 5 minutes, with four of them saying it would take exactly or on average 5 minutes. Twelve of the responses give the time range from 5 – 10 minutes. This includes one student who simply says: minimum of 5 minutes; another who claims: maximum of 10 minutes; one who specifies a range of 7 – 10 minutes; and three others who give 10 minutes as their exact answer. Three responses put the time range between 10 – 15 minutes.

The second response category was either a time interval of more than ten minutes, or two separate time intervals for easy and hard questions. In the larger time intervals there were four responses: 2 – 20 minutes, 2 – 30 minutes, a maximum of 20 minutes, and from 1 to more than 10 minutes. There were three responses that had different intervals for easy and difficult problems as follows: 1 – 2 minutes for an easy problem, 5 – 10 minutes for a hard one; 1 – 2 minutes for an easy problem, 4 – 5 minutes for a hard one; and finally 1 minute for an easy problem and 6 – 10 minutes for a hard one.

The final category was answers with no numeric value given. For nine of the students it was difficult for them to pin down an actual number for the amount of time they believed a problem should take. These responses can be further separated. For two of these complete novices, they did not believe it should take long to solve a word problem in general:

> *"Not long if you know the question's method."*
> *"No more than a few minutes."*

Because they have not given a time, we do not know exactly what is mean by "not long" and "a few minutes". However we can assume here that the time frame will be in the order of minutes. This is in sync with the previous responses and makes sense for these students.

At this stage in their mathematical career they would not have seen anyone working on a problem for longer than a few minutes. If they themselves are unable to solve the problem in minutes then they usually give up or seek help.

What is interesting is that the first response indicates a recognition that the amount of time is really problem dependent. This is also seen in the five following responses:

> *"Depends on the length of the problem and how many steps are necessary. No standard time."*
> *"It depends upon the complexity."*
> *"Depends on skills needed. Harder skills means more time."*
> *"I believe that it is unfair to set a given time for a word problem, as there can be so much variation between difficulty of each."*

These students may still have an idea that the time it takes is in the order of minutes, but realize that they can not put an actual time as this depends too much on the type of problem and the skills required to solve it. The fifth student quoted above not only refers to the time being problem dependent but goes even further to explain that it is "unfair to set a given time."

The final two students who responded in this manner both explained that the amount of time is however long it takes until you solve it. For the first student the idea is that the context in which the problem is given will determine the amount of time one should spend on it:

> *"However long it requires, depending, though on reason for answering them. On an exam, take time but don't take too long. If homework, do it until one is sure that one has no idea at all how to do it."*
> *"Until you solve it. There are still unsolved math problem[s] which have been around for 200 years. Maybe the semester should be that long. Just kidding."*

The second student makes an important point that there are problems that have been unsolved for centuries. This time scale is completely different from the other students; the first student provides no specific time frame, while the second student provides a time scale in the order of years. Both students explain that the time frame is problem dependent as it depends on the reason for solving the problem or whether the problem has been unsolved for a number of years.

## 7.2 Question 2: Initial Thoughts and Feelings

**What are your first thoughts/feelings when you see a word problem?**

This question parallels the second interview question given to intermediates and experts alike. What was noticed was that the experts had negative and positive feelings in almost equal measure to this question, while with the intermediates eight out of eleven of them reported negative feelings. The first thing to note with this question is that the complete novices' responses are very different from those interviewed. For the experts and intermediates, thoughts and plans are distinguished from the feelings. For the complete novices, the thoughts are often a reflection of their feelings. So we will separate this section into three categories. First we will deal with positive thoughts and feelings, then negative thoughts and feelings, and finally those who observe a transition in their reaction to first seeing a word problem.

**Positive Thoughts and Feelings**

While one student reported that his first thought upon seeing a word problem was "Yes! They are fun," for the majority of students who had positive thoughts, they were more along the lines of a plan of action for solving a given problem. Some of them asked questions: "What is the question I need to solve?", "What do I have to use?" or "How do I solve this?" Others spoke of trying the problem, whether it is trying to find familiar aspects, trying to understand or trying your best:

> *"I try to see what familiar aspects of the problem I know."*
> *"Understanding. Read through the question thoroughly and do your best to comprehend what is being asked."*
> *"Let's see if I can figure this out."*
> *"Try my best to solve it."*

This desire to identify the familiar aspects of the problem mirrors expert Sam Howison's first thought upon seeing a problem: "One of the things I always I suppose I always think about is, what do I know that's a bit like this? What have I done that's related?"

One student reports his realization of the difficulty of the problem while still having a plan on how to solve it:

> *"This will probably be difficult for me, but I should start with one step at a time."*

This comment is particularly intriguing coming from a complete novice as it reflects expert Colin Atkinson's view on dealing with problems: "I think you're worried and curious yeah. But you have to deal with it. You have to begin things that's how you progress."

Four out of the 45 students spoke of positive feelings explicitly. The experts expressed excitement, interest and curiosity upon seeing modelling problems and we note very similar feelings here. Students speak about feeling "excited and curious" as well as "challenge or intrigue". One student elaborates further on mathematics problems being compelling:

> *"I am compelled to solve the problem, I end up feeling self-gratified and happy that I am able to solve a problem."*

These positive emotions associated with solving mathematics affect people of all levels of expertise. Experts and complete novices alike feel the curiosity, excitement, the challenge of the problem, compulsion to get to the end, and then that happiness and satisfaction when the problem is solved.

### Negative Thoughts and Feelings

From this study we have seen evidence that there are modellers of varying expertise (from expert to complete novice) that have positive feelings towards modelling. Similarly negative thoughts and feelings are evidenced across the spectrum. Experts spoke of wondering if they could solve the problem, intermediates of being overwhelmed. As expected complete novices have a range of negative thoughts associated with their mathematics word problems.

The first theme we see here is that of confusion: "This is hard","I'm confused." This ties in nicely with the second theme raised by the complete novices of a lack of understanding:

> *"Is it one I understand? I hope it is..."*
> *"Hope I understand how to do it."*
> *"Uh oh, how should I begin this..."*
> *"My first feelings are of sadness. My first thoughts are of how I am going to do this."*

The use of the word "hope" in the first two above quotes may be misleading, as it speaks to despair and fear of not understanding more than it does to actual hope about the problem.

Not knowing how to begin mentioned in the third response above is also another big issue, which is related to the third interview question (see Sections 4.3 and 5.3). If students are unable to start then it is impossible for them to make any progress. We might also look at the last quote as simply a plan of action, but coupled with the feeling of sadness gives a strong negative feeling associated with the thought.

Other responses also convey a sense of dismay or dread upon first seeing the word problems:

> *"Oh no, not this again. Oh well."*
> *"OH NO!!! WHY CAN'T WE JUST BE STRAIGHTFORWARD!"*
> *"Argh. So much work and reading."*
> *"Yuck."*
> *"Oh crap."*
> *"The HORROR!"*
> *"I hate math/ this is going to take forever."*
> *"This is going to be hard - depends on difficulty or if I can see what to do right away."*

Expressed in these comments is the worry that the problem will be time consuming, that it's not straightforward, and that it is automatically hard, if the solution cannot be seen immediately. These are perceived as negatives because the expectation is that a mathematics problem is good, or a mathematics student is smart, if the problem seems straightforward and obvious. Many students also expect problems to be solved in 1–2 minutes as we have seen in the complete novices' responses to question 1. This is contrary to what the experts have demonstrated, where the interesting problems are those that are not straightforward (see Section 4.10).

Coupled with these thoughts are the twelve responses on negative feelings. Students speak of feeling everything from "Panic" to "Not happy" and "scared". Students go on to explain that they "hate word problems", and feel "confused and stressed" or "dreadful and confused". Some students elaborate further on the cause of their negative feelings:

> *"Annoyance, angst. Especially if I don't understand the question."*
> *"Nervous! Or I feel like I can't solve it. Word problems always seem impossible to solve when I first read them."*

> *"Panic, I'm not a fan of math and word problems tend to make me more nervous and confused, I begin to question myself."*

Again this is interesting because we know that the expert mathematical modellers express these exact sentiments:

> *"Um, well, feelings! That doesn't help! You know, how am I gonna understand this? I have no idea what the words mean. I don't know anything about this." (Sam Howison)*

> *"Feelings, it's kind of a joke but, I hate this problem for being hard, I hate my self for not being smart enough to know immediately how to write it down, or some variation." (Thomas Witelski)*

Finally one student explains that he feels numb when dealing with word problems:

> *"Numbness. There is no point in getting emotional. It must be done so do it."*

This is reminiscent of expert David Muraki's statement about his feelings: "I've learnt to suppress research anxiety" (Section 4.2).

### Transition of Thoughts and Feelings

Eight of these FAN X99 students describe a transition in their feelings or thoughts towards word problems. One student simply explains that it "varies" without any further clarification. Three others describe the change they go through when viewing a single problem:

> *"Ahhh ... No, wait – Oh, this is fine (now that I have read the question)."*
> *"Annoyance/Panic. Then curiosity."*
> *"Eek. Give me some time [and] I can do this. Have I seen something like this before?"*

These students describe a feeling of panic, which then turns into a more positive thought or feeling, whether it be a feeling of curiosity or a more calm state, or a plan of action. This is a bit different from the student who feels a difference depending on where the question comes from:

> *"In FAN X99, 'Let's do this!' In a difficult course 'Oh boy...' "*

For the final three students the transition has taken place throughout the course. They explain how they felt discouraged at the beginning of the course when seeing problems, but have now reached a stage where they are not as daunted:

> *"I used to think 'Oh crap' but after FAN X99 class I think 'Now I have to think more.' "*
> *"Before FAN X99 = discouraged to try. After FAN X99 = Willing to get an answer."*
> *"Depends on the question, for the first half of the course it was 'shoot me' and now I understand & enjoy them."*

## 7.3 Question 3: Initial Barriers

**What do you do if you have no idea where to start?**

This question is again a parallel to the corresponding interview question. Experts and intermediates suggest several different methods of dealing with the non-starting situation in modelling. These include trying to understand the problem better, talking to others, researching, just doing something, and solving a simpler problem. Experts also discuss taking a break and coming back to the problem when stuck.

For me it was unexpected that many of these complete novices expressed some of these expert or intermediate behaviours. Themes of trying to understand the problem, simplifying, and looking at related problems, just doing something and talking to others and taking a break were all raised. Students also discussed the use of drawing a picture as a heuristic to help them when stuck initially. A more in-depth look at each of these themes showed the difference in expertise in the way each of these themes was approached.

**Understanding the Problem: Re-read and Write**

Experts go about trying to understand the problem via research and discussion/collaboration. We will deal with the theme of collaboration later, but it is clear that reviewing the literature, *i.e.* looking at research papers, books and searching online, cannot be practiced in a classroom setting of complete novices. Four students instead spend time re-reading the question:

> *"Break question down slowly. Re-read question."*
> *"Read the question over multiple times."*
> *"I re-read the question, then think for a moment. Sometimes that works. If not then I go by trial and error. If it still does not work I ask for clues."*
> *"Try to re read the question and write down what I know."*

These are all good practices for understanding the problem. What we must note here is that for many of these students re-reading does not necessarily guarantee understanding. This explains why the question may need to be read "multiple times." To avoid this, the alternative strategy of listing the information given from the question is also used by six of the complete novices:

> *"Write down the objective and information I have."*
> *"Re-write the question in point form."*
> *"Listing out what I know (what the question tells me) and hopefully something comes to me. Or work backwards (guessing)."*
> *"Define facts about the problem. Think."*
> *"Pull out the information."*
> *"I list everything I do know."*

Note that "guessing" or "trial and error" is mentioned here, as a strategy if re-reading the problem or listing what is known does not yield a plan of action. These come under the heading of: just doing something, and will be addressed later in this section. Two students also mention that they "think." This is noteworthy, because of the tendency for students to rush to try to solve as opposed to taking time with the problem to ensure understanding (see Section 7.4).

The experts do not mention re-reading and writing out the given information. This does not necessarily mean that they do not practice this, but that it is not significant enough for them to note it. An expert would not consider himself stuck at the beginning if he had not first fully absorbed the problem and gone through it so that he is aware of the information given. This difference in perspective shows how complete novices view being stuck. There is an expectation of being able to understand exactly what to do at first glance. A complete novice can therefore consider himself stuck upon viewing the question once, if he is not completely sure of what the question is, or what information is given.

### Simplify

Another way to attack a problem is to simplify it. The theme of simplification was mentioned extensively throughout this work and is raised by experts and intermediates alike. As seen from the first quote in the previous section, "Break question down slowly. Re-read question," some of the FAN X99 students also recognized that this could be helpful. This simplification is described in two different ways. The first way is to attack the problem a bit at the time mentioned by two of the complete novices:

> *"Break up the question into parts [and] understand each part slowly."*
> *"I try to break down the main things to create an equation."*

Here by dealing with individual pieces of the problem these two complete novices can "understand each part slowly" or "create an equation." The second way to simplify the problem is by solving a simpler version of the problem which is suggested by three students:

> *"Give up. But then I realise if I break it down or try to solve a simpler problem it will help to get at least somewhere."*
> *"Write out all the information given in a simpler form."*
> *"I try drawing a picture or use smaller numbers to start."*

This last quote is most illuminating here, trying smaller numbers makes the problem easier for them to deal with or identify patterns or structure. Simplifying the problem shows a mathematical maturity and it is therefore not too surprising that such a small proportion of students mention it.

### Draw a picture

This heuristic is taught to the students taking the FAN X99 course. This is in correlation with the high proportion of expert mathematical modellers interviewed who mention drawing a picture as their number one means of getting entry into a problem.

> *"Draw a diagram."*
> *"I try drawing a picture or use smaller numbers to start."*
> *"Look at the numbers, draw a picture, put the numbers in different orders. Skip the question and come back to it later."*

> *"Leave it come back or break it down or draw."*
> *"Draw diagrams, search notes for similarities, brainstorm w/ classmates"*
> *"Try to draw a picture or write an equation."*

Notice straightaway that in all but one of these quotes, drawing is not the only thing that is attempted. Two students combine the theme of drawing with that of leaving the problem or taking a break. One other student suggests drawing along with collaboration and finally we see drawing as an alternative to writing an equation.

Even though this heuristic is taught, it does not seem to be one of the most used based on the fact that only six complete novices suggested it. This could be because the complete novices do not see drawing as a real option when they are stuck. This would also explain why one student responds that when stuck at the beginning he/she would "try and make a 'let statement' and then find an equation". This course of action was surprising to as it is unclear how one could do this if one was indeed stuck. But it again illustrates a desire to have an equation, which may be viewed by complete novices as having done some mathematics.

**Just Do Something**

Experts Colin Atkinson, Sam Howison and Lou Rossi mention starting somewhere, starting writing and trying things, when stuck initially. Six FAN X99 students also talk about just trying something if they are stuck at the beginning of a word problem:

> *"I re-read the question, then think for a moment. Sometimes that works. If not then I go by trial and error. If it still does not work I ask for clues."*
> *"I start by randomly using theories that I remember - if it's fractions, I will draw visuals, if it's a linear equation, I'll draw a graph."*
> *"Keep trying random ideas until they mesh."*
> *"Start somewhere and figure your way through it."*
> *"Honestly I just start with a random equation and hopefully I figure something out."*
> *"Trying out different things and maybe wait for what the correct way is."*

There is a difference between the experts who "start trying things" and the complete novice who claims to "start with a random equation" or to "keep trying random ideas". Experts are generally not working at random. While they may start somewhere that does not guarantee a solution, there is still a method and reason for their choices. This is not to say that the complete novices do not gain some measure of success from their work, but it must be rational or systematic to be considered expert behaviour.

### Similar/Related Problems

Solving similar problems is a viable heuristic mentioned by experts and intermediates alike that was also mentioned by four of the complete novices:

> *"Think of similar questions I have previously done and try to apply those to the question."*
> *"Think back to lecture, see what I learned, and figure out how they relate."*
> *"Try think of all the ways to solve the problem that I know and just meditate on it a while."*
> *"Draw diagrams, search notes for similarities, brainstorm with classmates."*

These similar problems are not always problems solved by the students themselves but may be problems that they have seen their instructor do. Once again we note that the majority of the FAN X99 students (41 out of the 45) did not choose this as a means of accessing the problem.

### Collaboration: Ask for Help

The theme of collaboration is slightly different in the case of the complete novice. Experts often speak of collaborating with others to clarify areas where they are stuck, as opposed to asking others to solve the problem for them:

> *"Otherwise talk to other people. They may not know how to do it, or be able to tell you. But just knocking around ideas. It's a reflective process." (Lou Rossi)*

Experts use language including terms such as collaboration, talking to others and calling a friend. In contrast complete novices often speak in terms of asking for help, and waiting for the correct answers:

> *"If I have no idea where to begin I ask for assistance of some sort."*
> *"Ask for help. Attempt to highlight key points."*

> *"Ask teacher or classmate."*
> *"Ask someone."*
> *"Wait for the teacher to explain or ask a neighbour."*
> *"Try plugging in what I can, if any, and then wait until someone else figures it out."*
> *"Skip it and ask someone later."*
> *"I ask for help but if I don't get it right away I do nothing."*
> *"Trying out different things and maybe wait for what the correct way is."*

This request for assistance is more passive than the term collaboration implies. There is a tendency to ask or wait for solutions methods from those they consider experts or having the answer. There are of course exceptions to this seen among some responses given, but these are a minority:

> *"I re-read the question, then think for a moment. Sometimes that works. If not then I go by trial and error. If it still does not work I ask for clues."*
> *"Draw diagrams, search notes for similarities, brainstorm with classmates."*

It could be argued that experts are behaving the same as novices when they ask for help from the person who brings the problem. And indeed to some extent this is true because the mathematical modelling expert is oftentimes a novice in the field of expertise of the problem and so must get some insight into the problem by asking for help.

However, there is a dynamic and an unspoken contract between the instructor and student that is different from that between the expert in the field and the mathematical modeller. In the classroom it is expected that the instructor, who brings the problem, already knows the answer. Even if they claim not to know, they have no real need of the students to get a solution. This is very different from the relationship between modeller and expert in the field. While the expert in the field is knowledgeable about the terms and the problem itself, they also view the modeller as an expert in being able to solve the problem. Thus there is a mutual respect of each other's abilities and the collaboration is more active.

**Take a Break: Move On**

Another suggestion by the experts if you are stuck initially, is to "wait it out" (David Muraki) or "sleep over it" (Reinhard Illner). We have discussed before that this idea of taking a break when stuck is modelled after Hadamard's incubation step in the process of mathematical creativity. However it must be preceded by the preparation stage, which requires serious conscious effort. When complete novices speak of taking a break, do they mean the same as the experts or are they simply giving up on the problem?

One student gives the impression that he/she makes an attempt to solve the problem, before leaving it and coming back to it:

> *"Look at the numbers, draw a picture, put the numbers in different orders. Skip the question and come back to it later."*

For two others, moving on from the problem seems to be a means of abating the anxiety associated with the problem as opposed to allowing time for the subconscious to work on it:

> *"Move on to the rest of the test and come back to it when I'm not stressed."*
> *"Panic, then move on to the next question and come back."*

From the first quotation we see that the student is assuming that the context for solving the word problem is in a test situation. It should be noted here that moving on from a hard problem in a test situation and coming back to it was a test-taking strategy taught to the students. Four other complete novices do not clearly indicate if any attempt was made to try to solve the problem first, or that upon re-visiting the problem if they typically have any more success than the first attempt:

> *"Skip to the next question and go back to it later."*
> *"Skip the question & go back to it later."*
> *"Keep trying or skip it and come back when all others are done."*
> *"Leave it come back or break it down or draw."*

While the above seven students say they will come back to the problem four others make no such claims. Their way of dealing with being stuck is to move away from the question altogether. The answer can be acquired through others or not at all:

> *"Move on."*
> *"Re-read the question, give up, cheat."*
> *"Skip it & move on to the next problem."*
> *"Skip it and ask someone later."*

For these last four students in particular it is clear that they have simply given up. The theme of moving on or skipping the problem like the previous theme of asking for help is a very passive way to deal with being stuck. It is therefore not surprising that these are the two most popular means chosen by the complete novices with each being chosen by eleven students. (Compare this to solving a simpler problem chosen by seven students, re-writing the problem, drawing a picture and just doing something at random chosen by six students each, and re-reading the question and exploiting a similar problem chosen by four students each). The tendency to wait passively for help or simply give up therefore aids us in defining the attitude of those who are inexperienced.

## 7.4 Question 4: Plan or Plunge?

**If you know what to do, do you plan your solution or do you "plunge" into it?**

In the FAN X99 classroom the questions given generally only contain the information relevant for solving the problem. Thus there was no point in asking a parallel question to interview question number 4 regarding how to determine what information was relevant to the model. However, of interest is whether complete novices take the time to plan their solution before putting it to paper.

Experts working on modelling generally are working in groups, and have to plan and delegate what will be done. Intermediates talk about recognizing that they should plan, but often opting to jump straight in first any way:

> *"Yeah when I started, it was just like, BOOM jump into the big problem cause that's what I wanna solve" (Liv I.)*

> *"A lot of time I just jump in cause I'm like, 'Oh I can do this.' Nope, nope you can't! Take a step back! So it would be a lot more effective for me if I didn't just jump in with both feet right away." (Isabel I.)*

Liv I.'s response indicates a level of excitement and a desire to solve the problem causing her not to stop and plan. Both of these intermediates go on to speak about learning patience with the problem as they have become experienced. This question about plunging into the solution helps us to identify if complete novices tend to jump right in without thinking the problem through, and their rationale for doing so.

### Plunge In

Of the 45 FAN X99 students that responded to the questionnaire, a majority of 28 explained that they plunged in once they knew what to do. Of those students many just responded by saying "plunge" or "plunge into it". But for many of them this was not enough. Some students emphasized their answers with exclamation marks: "Plunge!" one by using capitals only: "PLUNGE" and also, "Plunge in baby!" This shows a level of excitement in the response, which can be interpreted that this action of plunging in and solving is pleasing to do.

Other students go on to explain a little further about what they do when they plunge into the problem:

> *"If you know well enough, just plunge in."*
> *"I always seem to plunge into it if I know exactly what to do."*

In these first two responses the participants clarify that this action is taken because you know what to do.

> *"If I know what to do I do it as quickly and thoroughly as possible."*
> *"I plunge into it in an organised fashion! :)"*
> *"1) plunge into it 2) step back, review 3) try again 4) cross reference."*
> *"I mostly just plunge into it, though I usually keep in mind the steps I need to do."*

These four responses indicate that the participants are aware that plunging in is not necessarily the best way to go about solving the problem. There are clarifications about the "thoroughness" and "organized" way in which they do this, as well as speaking about reviewing and cross-referencing. The idea of stepping back, reviewing and cross-referencing, is expert behaviour, seen in Polya's [34] heuristic framework.

> *"I plunge into it by solving it out & seeing if it works"*
> *"I plunge into it and if it works, it works and if it doesn't I look back and see where I went wrong."*
> *"Plunge into it. Therefore I make stupid mistakes"*

These last three responses clearly indicate that the participants are aware that though they may believe they know what to do, they may still make mistakes by plunging in. This accounts for the "stupid mistakes" as well as the need to see "if it works", and where it "went wrong."

### Plan

Ten participants claim to plan even if they know exactly what to do. Here there was no use of exclamation marks or uppercase letters to convey enthusiasm. Most of these responses were simply: "plan." However four students explained their reasons for not plunging in without a plan:

> *"I plan my solution so to allow straight thinking"*
> *"It depends on the question, but planning a solution usually has the best results"*
> *"Plan out my solutions to get appropriate marks"*
> *"Plan/organize first to avoid making mistakes"*

These students have realized that by plunging in they are more likely to make mistakes, lose marks and not give a clear answer. However there is a caveat to always trying to avoid mistakes which is summarized by Liv I.:

> *"I think one thing that holds a lot of people back is not being able to jump in. Like when you're, when you're unsure of something, you're afraid to jump in. So I think one characteristic would be people who are good modellers are not afraid to make mistakes. And try, like just try something, you have an idea try something." (Liv I.)*

### Other

The remaining seven students did not give a clear answer as to whether they tend to plunge in or plan. Two students claimed to do "a bit of both". Two others explained they also did both but had a preference for one or the other:

> *"Both. Mostly Plunge"*
> *"A little bit of both, I plan as I am starting"*

The final three are more ambiguous with their answers and are probably the most accurate in describing their process:

> *"I don't really plan, but I try to write down all my steps as I go."*
> *"I do all the work to the side of the question and then write it in a more clear format."*
> *"I plan in my head while I plunge into it."*

What we do not see in response to this question is a description of planning, trying, evaluating, refining, and then evaluating again, as we would expect the experts to do.

## 7.5 Question 5: Organisation of work

**Do you feel your work on word problems is generally organized or disorganized?**

This is a follow up question to the previous one. Do students who plan consider themselves organized? Do the plungers think of themselves as disorganized? Here it is my aim to identify any correlation between questions 4 and 5.

Twenty-eight of the students described themselves as organized, while eleven of them said they were disorganized and six of them were unable to clearly state which word best described their work. As it was a follow up question, I noted that sixteen of the 28 that claim to be organized also said that they plunge into a problem as opposed to planning. Only seven of them were self-proclaimed planners. There were also more of those who claimed to plunge into problems in the group of disorganized students, with eight of them saying they plunge and only one student claiming to plan his solution.

There are several reasons and clarifications given on why the work is organized, including:

> *"I try to keep it organized so that my thoughts are easy to sort through."*
> *"Organized if I know how to do it."*
> *"It is generally highly organized, even if it is wrong."*

Similarly some of those who explained they were disorganized had further clarifications to make:

> *"It's all over the place because I use mental math to retain information. Therefore, my work will be everywhere."*
> *"Disorganized until answer is solved."*
> *"Disorganized but I'm good at them."*

There was also the group of students for whom it was not easy to categorize themselves on group or the other:

> *"I'd say depending on my understanding of the question."*
> *"If I'm running out of time = disorganized. Beginning of the problem = organized."*
> *"Depends if I know the answer. If it is a simple method, it is more organized. It becomes messy when I don't know right away."*
> *"I'd say it's improved since this course."*
> *"Fairly organized, though sometimes scattered."*
> *"Somewhere in between."*

The first two of these are self-proclaimed planners. This seems to fit with their descriptions of organization, particularly for the second student in this group. At the beginning of the problem he/she is organized and plans the solution. However from the data there seems to be no real correlation between a student thinking that he/she is organized and whether he/she plans a solution or plunges.

Upon further examination of the responses four interesting comments seemed worthy of discussion:

> *"My work I feel is generally organised (despite my sometimes messy handwriting)."*
> *"The actual work is disorganized but I rewrite it with more clarity."*
> *"My thought processes are organised. Written work unorganized but that because I'm generally a slob."*
> *"Organized because my writing is good/big."*

These comments about handwriting raise questions about the meaning the students ascribe to the term 'organized.' Which of them are referring to neatness of presentation, and which of them are referring to organization of ideas? This ambiguity in these students' understanding of the question makes it difficult to know exactly what any of the students mean when they say: organized or disorganized. This also highlights the fact that they may not be answering the question which is being asked.

## 7.6 Question 6: Dealing with Being Stuck

**What do you do if you are stuck in the middle of solving the problem? How does this make you feel?**

This question is designed to mirror the corresponding interview question 6 on being stuck. In the interview question I did not ask about feelings as I had already dealt with the feelings surrounding being stuck in the second question directly. However I thought that for the complete novices several opportunities might be necessary to talk about feelings, this was reinforced by the way feelings and plans were tied in the responses to the second question above (see Section 7.2).

Experts and intermediates alike discussed several means of dealing with being stuck in the middle of the problem. These were collaboration, research, simplification, applying a different approach including starting over, taking a break as well as persistence. The complete novices that responded from the FAN X99 class spoke to several of these themes, which will be discussed below.

#### Feelings

The experts were not forthcoming with their feelings when asked. The intermediates required less prompting, but often seemed surprised at being asked about their feelings. It should be no surprise then that the complete novices generally spoke about their feelings, sometimes at the expense of explaining what they do when stuck in the middle of solving a problem. There were however still 5 students who did not discuss feelings, only their strategy for becoming unstuck.

The overall feeling associated with being stuck is one of frustration. Fifteen students said that they were frustrated when stuck. Others explained that they felt annoyed, panicked, bad, angry, low in confidence about their mathematical ability, stupid, nervous, uneasy, stressed, horrible and unsure, or to put it in the words of one student:

> *"I don't like not knowing how to do a question"*

I will look at all other quotations about feelings in conjunction with their associated plans for getting unstuck.

**Taking a Break: Move on and come back or give up?**

Sixteen students chose the option of moving on to another question, or taking a break. Some of these students chose this option after trying to go over the problem:

> *"Start over and if it still doesn't work I generally give up. Makes me feel SO frustrated."*
> *"I double check my previous work, something must have gone wrong there. If not, then I move on."*
> *"I get frustrated & possibly move onto the next question or start over & check my work for mistakes."*

But for sixteen of them the only option mentioned is to move on from the problem:

> *"If I get stuck I move on. I feel bad because I don't like giving up or saying 'I'll come back to it'. I think that is an excuse."*
> *"Skip and go back to it later. Frustrated."*
> *"Come back after. Sometimes I over-think."*
> *"I go back to it later, it makes me feel stupid."*
> *"It's annoying, so I leave it and do something else for a bit."*
> *"It makes me feel horrible! I just leave it, go through the rest of the exam and return to it in the end."*

Note that for the last student quoted above (and possibly others), the context is again assumed to be a test situation, where they have been taught to move on and come back as a test-taking strategy. For these above students the intent is to come back to the problem. But as discussed in Section 7.2 moving on can mean giving up on the question altogether, at least for seven of these students:

> *"It makes me feel frustrated and I want to give up."*
> *"Uneasy. Nervous. Solve a different problem."*
> *"Take a break and go back to it. Arg!"*
> *"I usually move onto the next one and come back to it. It makes me slightly worried."*
> *"I usually stop working on it. It makes me feel frustrated."*
> *"I feel frustrated when I hit a wall. I usually get upset & just stop working all*

> *together."*
> *"I get frustrated and move to another problem."*

This is a striking difference between those with more expertise and the complete novices. The option of simply giving up is never mentioned by the intermediates or experts. This can be attributed to ability and experience, as well as the situation in which experts or novices are dealing with the problem. For the experts in particular, any problem that they choose to work in is one of interest. This is very different from the situation of these complete novices who have no choice in which problems they solve.

However, intermediates may find themselves in a similar situation of solving problems not of their choosing or liking and they also did not discuss the possibility of giving up. This can again be attributed to the difference in their situation. Even though an intermediate may not choose the problem, he/she is aware that doing the problem is in their best interest. For the complete novice often the problem is difficult, not of interest, it is not of their choosing and they do not see how it is useful for them to be able to do it. It is therefore viewed as a futile exercise.

#### Go over the problem

One of the themes raised by the experts in Chapter 4 is that of starting over when stuck. Fourteen of the FAN students described starting over as their means of trying to become unstuck. For five of them, this entails re-reading the question and looking through their steps:

> *"I go back to the beginning and re-read the problem to see if something will click. It's frustrating."*
> *"Panic then look through my steps and re-read the question. I feel unconfident."*
> *"Re-read a question. Feel nervous."*
> *"I retrace my steps"*
> *"If I can't figure it out after going back and taking a 2nd look, I usually go into a rage. Very angry."*

There is no indication here what the purpose of looking over the steps is or what re-reading the question will do. One can assume that there is a hope that by re-reading the question they will get new information. This is similar to researching, or going back for more data as described by the experts, although it is not articulated here.

Those who claim to retrace their steps do not express that there is an expectation that they are doing anything different. Other complete novices explain that they start over completely:

> *"Start over and if it still doesn't work I generally give up. Makes me feel SO frustrated."*
> *"I start over, or I get myself unstuck."*
> *"If I get stuck I go back to the beginning, and work it through again. But I would be annoyed."*

This is more useful than simply retracing steps, as one may not make the same mistake twice. Another way would be to go through the problem again looking for mistakes being made:

> *"Annoyed. I take a moment to think through my steps, both before and what to do next. I also check to make sure I did the previous steps correctly. I also re-read the question."*
> *"I get frustrated and then breathe. Afterwards I keep the work I have down and look over it to see where I could have gone wrong."*
> *"I double check my previous work, something must have gone wrong there. If not, then I move on."*
> *"I get frustrated & possibly move onto the next question or start over & check my work for mistakes."*
> *"Start over to see if I made an error along the way. Frustrated."*

However this does not account for the fact that the initial method may be completely wrong. For the expert, starting over does not mean simply, solving the problem the same way a second time, but re-thinking assumptions and solution method. This was only articulated by one of the FAN students:

> *"I go back to the beginning and try again. If it still doesn't work I try it again in a different way. It bewilders me mostly."*

### Collaboration: Ask or wait for help

Collaboration was one of the main methods used by experts and intermediates alike when stuck in the middle. Experts go back to the one bringing the problem, or simply collaborate with their colleagues. Intermediates often talk to supervisors, or others within the group. What is surprising here is that only four out of the forty-five FAN X99 students offered this as a solution for being stuck in the middle of the problem:

> *"If I'm in class, I will wait until we go over the problem, otherwise I ask a friend or even 'sleep on it' if I have time. It makes me feel like I am poor at math."*
> *"I ask. Makes me feel like I need to study more."*
> *"It annoys me! I usually ask for help from someone around me or the prof."*
> *"I try to keep going, if I cannot do much more, wait for the correct way. I feel unsure."*

This is also in contrast to the eleven who recommended this plan of action if stuck at the beginning. Indeed, eight of those eleven students who would ask for help if stuck at the beginning of the problem, do not choose this option if stuck in the middle. This contrast highlights the fact that, at the point where experts deem collaboration the most useful, complete novices seem to choose to avoid asking for help or even waiting for help.

### Other

The FAN X99 complete novices raised a few other themes. The first one that we will look at is persistence:

> *"I would keep trying no matter how frustrated I become."*

Hadamard explains that continuous persistence is useful to a point [32]. The novice does not explain if he/she would keep trying the same method that is not working, or several different methods. The theme of trying something different is raised by two other students:

> *"Try new methods. Frustrated."*
> *"Try something else. Like any other time you get stuck on something."*

The second quote above reveals that the student recognizes that the skills required in mathematics class parallel those in many other situations.

Other students mention trying to identify where the mistake could be, working backwards, and refocusing, as strategies when stuck:

> *"I try to think where I may have gone wrong and then proceed to slowly try and fix it. It makes me feel tense and at times angry at myself."*
> *"I'll work backwards [and] see where I went wrong. Flustered."*
> *"Happy of course. I love coming up short. Jokes. I usually get a bit frustrated then refocus."*

Having several methods for getting past barriers in an open mathematics problem is vital. Experts and intermediates often mention several that they have found useful. These complete novices more often report only one thing that they do when they are stuck. Most importantly, at this stage of the modelling process, experts seek active collaboration with each other more than any other strategy. This is lacking in the complete novice responses.

## 7.7 Question 7: Problem Solving Heuristics

**What sorts of problem solving techniques do you use?**
**(eg: Draw a picture? Work backwards? Solve an easier problem?)**
**Use specific problems from the course to illustrate if you can.**

This question was aimed at identifying the heuristics used by these complete novices on word problems in class. I once again only gave three in order to avoid overwhelming the students and to be more consistent with the other data gathering questions.

### Draw a Picture

An overwhelming 36 of the FAN X99 students (80%) cite diagrams and pictures as one of their preferred heuristics. This seems to be especially the case for, but not isolated to, dealing with fractions:

> *"I found drawing a picture to be very useful when dealing with percentage problems and finding fractions of fractions."*
> *"Drawing a picture makes the problem easier to understand [gives a drawing of a pole with the caption: a pole is pushed some % into the ground how far...]."*
> *"I always draw pictures (like the monkeys in the tree problem or the swimming*

> *pool problem)."*
> *"Percentages = charts/pictures = visual. Visual diagram works best."*
> *"I look at the numbers, get a feel for the range of the answer. Then draw a picture and figure it out visually."*
> *"Whatever I need. Diagrams can be helpful. i.e. "You cut 3 equal pieces off a wooden plank 180 cm long. Develop a formula." I'd draw a plank then do something like [divides rectangle, marked 180, into 4 pieces where 3 are equal size and labelled x] = 180 - 3x."*
> *"Drawing pictures really helps. Like graphs or fractions. Example [draws 2 circles divided into quarters with 5 quarters shaded]."*

First of interest is the fact that for 12 of these 36 students, drawing a diagram is the only heuristic that they mention. Secondly, many of them are able to give specific problems and make use of illustrations to show how they would use pictures and diagrams to solve these problems. This shows as clearly as the number of students, how fundamental drawing pictures is for them. This may seem at odds with the fact that only 6 of them mentioned drawing a picture when stuck. However, this may be a technique not reserved for being stuck, but used before being stuck on the problem. We also cannot discount the possibility that of the three heuristics mentioned, this might simply be the preferred one.

### Work Backwards

Six of the forty-five FAN X99 students that participated spoke about working backwards. This skill is a little more sophisticated, as it requires recognition that it can be used in a particular situation. Note that working backwards is not mentioned by itself in any of these situations, as it is very problem dependent:

> *"Sometimes I'll draw a diagram or work backwards, it depends on the problem."*
> *"Pictures always help. Rewrite key points. Cross ref. Reverse order."*
> *"I draw a picture, and label. Sometimes I work backwards. Often I then try to solve it algebraically if possible."*
> *"Draw a picture, try to work backwards. i.e. The problem about who did the fencing in the garden I tried to map it out [illustration of a square divided up several times]."*

> *"Questions about mass and/or volume, I draw a picture. Questions about percentages, I work backwards (ex: starting with the result). Questions about graphs/functions I draw a graph if I can or list values. In most other cases I draw a basic diagram and/or solve an easier problem."*
> *"I use all sorts of techniques:*
> *-illustrations,*
> *-equations,*
> *-working backwards."*

However many of the experts did not cite working backwards as being a preferred skill either. And one FAN X99 student explains that he rarely uses this heuristic:

> *"I often draw a picture or solve an easier question. I hardly ever work backwards."*

**Solve a Simpler/Similar problem.**

Simplification is a major theme revisited many times by experts and intermediates as well. FAN X99 students also touched on solving an easier problem, as seen in the previous quote, as a means of getting unstuck. Nine FAN X99 students speak to it as one of their problem solving strategies:

> *"I draw pictures, solve easier problems, do anything to avoid long division. More but hard to remember unless faced with a problem."*
> *"I use all those techniques listed but I find memorizing an easier problem helps most because it's easy to go back to the basics."*
> *"Draw pictures & sometimes solve easier problems."*
> *"Sometimes I use drawings to help but I do try & solve easier problems first."*
> *"Picture [and] the easier problems within the big problem."*

Once again notice that this heuristic is not isolated. Drawing a picture is often another option used by these students. Another way to simplify the problem is breaking it down into easier parts:

> *"I can use diagrams sometimes, but I like to break down the main components to create an equation."*

> *"I usually do the problem in pieces and then put it all together."*
> *"Break up the problem and try to solve."*

Two other FAN X99 students spoke of looking at a similar problem as opposed to a simpler one:

> *"Look at past problems that are similar."*
> *"I draw, or draw from previous similar questions."*

These students are using their experience in order to deal with future problems. This is also not so straightforward as one has to recognize which problems are similar in solution method. Often problems that require the same strategy look quite different in terms of the language used. In order to truly understand if problems are similar, one must understand the underlying structure of the problem. From the literature [18] we know that novices tend to focus on surface features as opposed to underlying structure.

### Other

Many other techniques were mentioned. For some of them algebra or equations were mentioned as problem solving techniques. This illustrates that these students do not realize that algebra is a mathematical technique, but not a problem solving technique *per se*.

One student says that his/her problem solving strategy is "mental math" again illustrating his/her lack of understanding of the difference between a problem solving technique, and mathematical technique:

> *"I have decent mental math so I use that more than anything. If I'm really stuck I'll draw a picture"*

Another student relates problems to real life in order to solve them:

> *"Draw a picture, relate it to real life and put myself in the situation."*

Five students claimed solving the problem step by step, systematically or using logic was their method of solution:

> *"I try to work in a very linear manner and go step by step. I rarely, if ever use pictures."*

> *"Write the question down. Go step by step. Define the variables. Write down all relevant information."*
> *"It really depends on the question. If I can I like to solve things with an equation or systematically. If I have no idea where to start, I draw a diagram."*
> *"Find an equation [and] solve algebraically. I don't find picture as helpful, unless I'm double-checking my work."*
> *"Logic. Pictures."*

To solve a problem systematically or logically is never mentioned by experts as these are a given: this is an expected behaviour when dealing with a mathematical problem. We would expect the experts to mention only the extra-logical as this would be worth mentioning to them. Here we get an understanding that for these students, this is a significant process and thus worth mentioning.

Finally there are two students whose response gives no indication of any problem solving techniques being used at all:

> *"I don't use any techniques, If I know it then I know it."*
> *"I try to work with what I have."*

## 7.8 Question 8: Verification

**How do you check if your solution is correct?**

This question parallels the corresponding question 8a) in the interviews about how to check if you are pursuing a sensible solution. The experts mentioned several methods, including predictions, intuition, comparing the solution with data, looking at special cases, checking that the model makes sense, looking at the qualitative behaviour of the model, and comparing two methods. Intermediates also added use of convergence testing as a means of checking results.

Here I was expecting that the FAN X99 students were more likely to find out their work was correct by going to their instructor. In this assumption I was incorrect as FAN X99 students mentioned substituting the answer into the original equation, checking that the solution made sense, working backwards, re-calculating as well as checking with others.

### Substitute the Solution into the Equation

This was by far the most popular method for checking whether the solution was correct, with 25 students claiming that this was what they would do:

> *"Sub it back into the question."*
> *"I plug numbers in x to make sure it can be solved. Look rationally at the question and my solution."*
> *"By inputting it back into the question."*
> *"I input my answer into the original equation."*
> *"If it is a question like 'solve for x' I will sub the number I got as x back into the equation and see if it works."*
> *"I re-work everything into the original equation."*

Based on the types of questions that these students encounter, this is one of the easiest ways to check if they have the right answer, provided that the equation is correctly expressing the problem. Note that this check does not guarantee that they have modelled the problem correctly with their equation to begin with.

### Check that the Solution "Makes Sense"

Experts and intermediates alike raised this theme. The model needs to be internally and qualitatively consistent. Thus the fact that these complete novices also attend to whether the solution makes sense is an improvement on simply re-substituting the solution into a possibly erroneous equation:

> *"Sub in numbers and check to see if the number makes sense."*
> *"I plug numbers in x to make sure it can be solved. Look rationally at the question and my solution."*
> *"Plug answer into variable. Think logically 'Does this make sense.' "*
> *"See if it fits back into the equation or question and makes sense."*
> *"Sub my answers in to the question. Reread with my answers. Does it make sense?"*

Some students combine checking if the solution makes sense with other techniques including solving the problem more than one way and referring to the original picture:

*"I look at my picture or I check it algebraically. I also check to make sure my answer makes sense to me, given the question."*
*"See if it makes sense. Do the problem over using a different method."*

Eleven FAN X99 students in all speak about the solution "making sense"; many of whom do not clarify what they mean by the term "make sense" and simply state it:

*"Does it make sense?"*
*"I review it and see if it makes sense."*

However, two students are more explanatory in their responses:

*"If I can plug in numbers I do so. Otherwise I check my work again to see if I made any mistakes."*
*"Run through the calculation again. Check positives and negatives."*

One student is looking for mistakes in calculations. The other is looking at the qualitative behaviour of the solutions: are the numbers negative or positive where they should be? This is more sophisticated behaviour than his/her colleagues and is mentioned by expert Sam Howison as something to look for in order to eliminate mathematical errors:

*"There're the obvious ones like is the answer positive when it should be? But that's just to eliminate mathematical errors." (Sam Howison)*

**Work Backwards**

This technique was raised by nine of the FAN X99 students who said that this was one of the ways they checked the accuracy of their solution. This technique is surprising in light of the fact that the experts and intermediates tended not to use this to check their work at all. However the literature [22] informs us that novices tend to work backward. This may also be because this heuristic was raised in the previous question:

*"Depends on the question. Usually I'll work backwards, or put my information into the original problem"*
*"Work backwards/ plug in original formula with the solution."*

*"Work backwards. Pictures."*

*"I work it backwards if possible."*

*"Work backwards."*

*"Cross reference and reverse order."*

*"I work backwards from the answer."*

*"Do the problem backwards."*

Of note here is the fact that only three of these nine students mention working backwards in conjunction with another checking technique. Also only two of the nine mentioned working backwards in their discussion of problem solving techniques used, suggesting this is only used to check answers not to solve them for the other seven.

**Re-calculate**

Another means of checking if the solution is correct is to re-do any calculations:

*"Run through the calculation again. Check positives and negatives."*

*"Skim over each step I took. Or plug in my answer."*

*"Plugging it back into the equation or just try it again later."*

*"Re-do the problem, or sub in x value to check."*

While this could help catch calculation errors, a misunderstood concept would simply be repeated. This makes this method one that is not a reliable check if the solution is correct, unless in the case of the last student quoted above, "re-doing the problem" involves more than re-doing calculations. Another student acknowledges the need to go back to the original problem to check the solution, in his response:

*"Going back to what the question is asking originally."*

He checks initial problem, as opposed to simply looking at mathematical errors. This is expert behaviour (see Section 4.9). Expert Thomas Witelski also suggests re-calculation using an independent method, which was mentioned by one of the FAN X99 students:

*"See if it makes sense. Do the problem over using a different method."*

This may be more helpful in identifying a fundamental mistake in one of the methods. However, once again if the original interpretation of the question is incorrect then both solutions may be incorrect.

### Check with others

Only four of the forty-five FAN X99 student participants suggested checking their answers with others:

> *"I would plug my answer in to my equation to check if my solution is correct (and then refer to an answer key if applicable)."*
> *"No real method, sometimes I see what other people answered."*
> *"I usually go back and plug my answer in to the question and when I'm lazy I wait until the prof does the answer."*
> *"Either check back (in the case of Algebra) or verify with an instructor or tutor."*

Interestingly, only one student mentioned checking with others as their only means of verifying the accuracy of the solution. This is surprising as I expected this to be the most popular means of verifying the solution at this level.

### Other

Three of the FAN X99 students responses illustrate that they did not fully understand the question:

> *"Yes."*
> *"Depends how much the question is worth usually more than 3 marks I do."*
> *"Most of the time."*

These responses to the question "How do you check if your solution is correct?" serve to highlight one of the main differences between the expert and the complete novice. The complete novice will not always take the time to ensure that he/she fully understands the question before answering.

Finally we note that two FAN X99 students explain that they do not check their answers:

> *"I usually don't."*
> *"I never fully complete them... so I don't check."*

While experts explain that you must check your answers, these complete novices show that this is not something that they do.

## 7.9 Question 9: Difficult vs Easy

**Give an example from the course of a word problem you found difficult and one you found easy. (Can you explain why?)**

When asking this question I had my preconceived ideas of what these complete novices would find easy and difficult about a problem. While with the experts I expected a discussion on the complexity of the problem structure, with the complete novices I fully expected them to complain about the length of the equations themselves and the mathematics involved.

The experts discussed more than the complexity in the structure of the problem. Themes of mathematical difficulty, familiarity with the problem and the clarity of the problem also emerged. Intermediates also identified poor collaboration as an aspect that made it difficult to solve a given problem.

For thirteen of the FAN X99 students there was no response to this question or they said that they were unable to remember any examples. Another three students did give examples but failed to indicate whether these were considered difficult or easy problems for them. This again illustrates the likelihood of the complete novice to misunderstand or not answer a given question in a meaningful way. However there were still several problems and subject areas listed as being difficult or easy. We will look at each category separately.

### Easy

The most popular topic considered easy by FAN X99 students is algebraic equations. This includes systems of equations. Students also explain that skills or quiz questions (which were also purely skills questions as opposed to word problems) were considered easy. Other students preferred percentage problems, or fraction problems:

> *"Simple problems for me include writing algebraic equations for word problems."*
> *"Ones with setting up an equation "let x equal" were generally all easy for me."*
> *"Generally, I found the problems where we needed to find an equation easier."*
> *"Easy: system of equations."*
> *"I found problems with one or two unknowns easy."*
> *"Easy = skills testing, or plugging in equations algebraically, or fractions."*
> *"Eq'ns involving fractions (and percent) = Great."*
> *"Easy: Fractions [and] graphing."*

> *"Ones where pictures work are easy" [includes a drawing of a picture of fractional amounts].*
> *"Easy - percentages and money. I don't remember specifically."*
> *"Almost all problems were easy."*

These topics give us some insight into what topics students seem to enjoy but do not explain why. Six students also explained why they preferred particular problems to help us to understand some of their reasons for picking their chosen easy problems:

> *"Quiz questions are always easy, very straightforward."*
> *"Easy would be those algebraic questions, as they are more rule/formula based, though I understand the reasoning behind my answers."*
> *"Easy: percentages, systems of equations. Why? Personal skill."*
> *"An easy one was the Tweedle Dee and Dum weight equation because I like simple 2 question equation[s]."*
> *"Easy - algebra (because I know it well)."*
> *"Well there are the easier algebra problems. i.e. $56 = 8x$ because I know my times tables very well."*
> *"The one about carbon emissions was easy (I like percentages :) )."*

Reasons mentioned for describing these problems as easy are: personal skill, liking the topic, the problems are straightforward, familiarity with the topic, and the problems are formula based. These reasons mirror those of the experts. Straightforward or formula based problems speak to problem clarity. The experts mention familiarity with the topic explicitly. Personal skill also implies a familiarity with the problem. While having a liking for the problem explains why it is seen as easy, we do not know why there is a liking for these problems as opposed to others. While these students do not explicitly speak about the structure of the problem here, we do note from the examples given that the problems that they tend to like appear conceptually straightforward to deal with.

### Difficult

The topics and problems considered difficult were more varied than those considered easy. Students discussed topics including: functions, graphing, rates, questions without clear or specific numbers, percentages, distance, area, and fractions. They also listed several specific problems that they found difficult:

*"Difficult - A LOT!"*
*"Graphing anything is difficult."*
*"Equations with graphs = NOT FUN."*
*"Fraction problems were difficult."*
*"Harder questions would be word problems with graphs and such."*
*"One difficult problem included someone paddling upstream a certain distance at a certain speed, and then downstream at a different speed. We had to find how far he paddled both ways to end up at the starting point in one hour. I couldn't figure it out."*
*"Difficult: Questions involving time/rate/pace of work with multiple people."*
*"Hard: solving word problem, word question. Eg locker problem at the beginning of the year."*
*"There was this hard one to do with monkeys in a tree asking how many were on the ground...I still don't understand it."*
*"Hard: the amoeba problem and lily pond."*
*"Difficult: adding consecutive numbers: 1, 2, 3, 4 (like the dart question or shaking hands questions)."*

First note that in the dartboard problem there is no need to add consecutive numbers. But also note that there is a greater number of specific problems that are mentioned here. It was difficult to find this many specific problems that students found easy. This implies that the difficult problems have a greater impact and are remembered better than problems that students considered easy. But exactly what about these problems make them difficult? Once again these above-mentioned students have failed to explain why. However, seven students did manage to give their reasons for why they found these particular problems difficult:

*"Functions are hard, hard to conceptualize."*
*"The gold rings problem near the beginning. I didn't quite understand the process of solving word problems."*
*"A difficult one was finding the percentage of weight from the kilograms of a tomato. I had trouble with conversion and it was a lot of equations."*
*"Hard - the most recent thing we're doing (because I get mixed up with domain and range)."*
*"Difficult = Word problems (logical thinking)."*

> *"Word problems in equations because I need to work on my skills."*
> *"Too many word problems. Everything had challenges from method of math or arrangement of words in a sentence."*

Reasons given for finding the problem difficult are also varied: difficulty conceptualizing, having problems with the process of problem solving, conversion and the number of equations, unfamiliarity with the terms, lacking logical thinking, lacking other skills, and the number of words in the actual problem. Issues with the number of equations, and the number of words in the problem, speak directly to the apparent difficulty of the mathematics as opposed to the structure of the problem as expected. Difficulty with concepts, logic and an understanding of the problem-solving process are less superficial issues. These complaints illustrate that the students are lacking basic tools to be able to solve the problems given. What makes the problems difficult is not only the difficulty of the mathematics, but the deficiency in ability of the problem solvers.

## 7.10 Summary

The focus of the FAN X99 class is not mathematical modelling *per se* and so the students were not asked a definition of mathematical modelling. When asked about the time frame for solving modelling or word problems, most students stated that these problems should be solved in the order of minutes. However, two students explained that the time taken to solve the problems are problem dependent. Students were very forthcoming about their feelings when solving word problems, for the most part expressing fear, dread, panic and anxiety, although several of them also expressed feelings of interest.

It should be noted that these students have little to no autonomy. For many of them the course is compulsory for their degree, and in class they do not get to choose which problems they prefer to work on. Unlike the experts, these complete novices tend to find difficult questions frustrating as opposed to interesting, with the theme of frustration being evident throughout most of the students' responses.

The complete novice participants tend to plunge rather than plan their solutions, but the majority of them thought of themselves as organised. The suggestion of giving up completely when stuck was only mentioned by members of the complete novice participants. Interestingly, several complete novices discussed expert heuristics such as simplifying the problem and drawing a picture to get access into the problem. Also of interest was the transition for some of the students, where they noticed that they became less anxious and more willing to try to do problems by the end of the semester.

# Chapter 8

# Conclusions

Throughout this work I have summarized my findings chapter by chapter. I now present an overall conclusion of all findings. In this chapter I first review the findings in the literature, highlighting the issues that pertain to this work. Finally, I answer the research questions that emerged from the lack of coverage in the literature, using the data from this study to do so.

## 8.1 Contribution to Mathematical Modelling

From the literature on mathematical modelling it is apparent that there are certain gaps and a need for extension in some areas. The literature describes the cognitive and metacognitive aspects of mathematical modelling, but there is no evidence of the extra-logical processes (creativity, intuition or illumination) or the psychological aspects. I therefore set out to get a more concrete understanding of exactly what mathematical modelling entails. I was particularly interested in the process of moving from being stuck to unstuck. I investigated the landscape from novice to expert at the tertiary level, looking not only at cognitive processes, but also exploring the feelings experienced when modelling, the psychology of the participants and the extra-logical processes required. What follows is a discussion and summary of these findings.

## 8.2 Research Question 1: What IS Mathematical Modelling?

We have seen that the literature describes modelling in terms of the cognitive processes, using flow diagrams to describe the process of modelling. This is not enough as the experience of modelling is not a simple traversal along a flow diagram. It is important that we have a more thorough understanding of what is involved in the mathematical modelling process. To paraphrase expert David Muraki, we have to know what we are talking about to talk about it (Chapter 4). We also note that in the literature there are two different approaches to teaching modelling: case studies and modelling by design [18]. However, no justification is made for this dichotomy in approach. Further, the Dreyfus model [46] indicates that learning via case studies is a skill that requires almost expert competence.

In Chapter 4 the first feature that emerged from the interviews with the experts was a dichotomy in the definition of modelling. Some experts saw modelling as a description: a way to describe a real world situation using a simplified mathematical construct. Others defined modelling as a process, which started with a description of the real world using a mathematical framework, was followed by the solution of the mathematics, and finally returned from mathematics to the real world. This dichotomy in definition may explain the dichotomy in approach towards teaching mathematical modelling.

The second idea that emerged was that the problems in mathematical modelling and problem-solving differed in reasonableness in the Perkins sense. Perkins speaks of unreasonable problems being those that cannot be approached by deliberate conscious effort. This is not a requirement in the definition of mathematical modelling as a modelling problem can be straightforward from the beginning to end. However, in response to question 9 (see Section 4.10), the interesting problems for the experts are those for which are not straightforward:

> *"A [modelling] problem is interesting when the known recipes don't apply and then you have to develop your own toolbox for it." (Reinhard Illner)*

Thus while problem-solving techniques are not a requirement for all modelling problems, they are useful for dealing with those modelling problems that are "interesting." As we will see in the next section this is evidenced in the responses to being stuck, which include extra-logical processes.

## 8.3 Research Question 2: How does one move from stuck to unstuck?

The mathematical modelling literature does not give a satisfactory description of this complex step. The term "simplify" assumes that the modeller will know how to do so, but does not address the fact that the modeller may be stuck because they are unsure how to simplify their model any further. Problem-solving literature reveals when someone is stuck because the problem cannot be solved by reason (*i.e.* it is unreasonable in the Perkins sense) that deliberate effort is no longer the key. This is the stage in mathematical creativity where one must wait for illumination or an AHA! moment [32]. These moments can be occasioned [33] but not forced. Considering we have already established that mathematical modelling problems do not by definition have to be unreasonable, this problem-solving approach to getting unstuck may therefore not be a viable method for the mathematical modeller.

The modelling experts of this study recommended several strategies, key among them are communication and collaboration whether stuck at the beginning or the middle of the problem. This theme is not recognized in the literature as a strategy for getting unstuck. Communication with the person who brought the problem, colleagues working on the problem alongside them, or even others who were not involved at all, were all recommended by the experts. This theme was not isolated to experts as intermediates also discussed with their supervisors and others around them when stuck. The theme of collaboration also highlights the fact that for many of the expert participants, mathematical modelling is a group exercise.

The theme of simplification raised in the literature was seen in the responses as well. This included looking for simple examples, removing modelling details, solving a small part of the problem, or a related trivial problem. Simplification was often used in context of trying to understand the problem better. Other themes were raised as well to help transition from being stuck to becoming unstuck. Experts spoke of looking at others' previous work to deal with being stuck at the beginning of the problem (see Chapter 4), and trying something different (different thoughts, different model or modelling assumptions, or a different approach) when stuck in the middle of the problem (Chapter 4).

Experts also described taking a break, waiting it out, and going for a walk when stuck, mirroring the discussion by Hadamard about illumination after incubation (taking a break)

[32]. This illustrates that although mathematical modelling is not identical to problem-solving, the extra-logical aspects of problem-solving are useful when dealing with the issue of being stuck.

Finally one of the most important themes that emerged is that of understanding why you are stuck. Experts spoke about trying to understand the problem by discussion and simplification. They often clarified that it depended on why you were stuck, whether you were stuck on the math or the modelling aspects and how you defined stuck (see Chapter 4), which determined if communication, trying a different approach or taking a break was the way to go.

## 8.4 Research Question 3: What are the differences between the novice and expert modeller?

In the literature there is evidence of several differences between experts and novices; however, these differences refer to the cognitive and metacognitive ability of the modeller (Chapter 2). I now look at a landscape of modelling ability from novice, through intermediate, to expert, highlighting the key differences in psychological and extra-logical processes as well as looking at the cognitive and metacognitive aspects of modelling. I first revisit the theme of being stuck, while comparing the responses of the other groups to what we have already seen from the experts. This is followed by a discussion of the psychology of the varying levels of modellers gleaned from their responses to questions 2 and 9. I then embark on a journey from the beginning of the model (responses to questions 2 and 4), through the skills deemed important (responses to questions 5 and 7) and finally look at verification of solution (responses to question 8a)).

### 8.4.1 Dealing with being stuck

In the previous section we have already discussed how experts move from being stuck to becoming unstuck, using collaboration as a primary resource. Other themes raised in response to this question were: looking at others' previous work, trying something different, and taking a break. For these experts, the key aspect of transitioning from stuck to unstuck is understanding why they became stuck.

The three main themes mentioned by intermediates when stuck were research, collaboration and taking a break. In contrast to the experts, research was the number one solution chosen when stuck, with ten of the eleven intermediates citing this, especially at the beginning of a problem. Seven intermediates also mentioned collaboration with colleagues or supervisors, but this was often after research had been done (Section 5.3). The theme of taking a break was mentioned as well by five of the intermediates interviewed. One intermediate mentioned taking time to think, but this is qualitatively different from taking a break, or the incubation stage, where conscious effort ceases. Trying to understand the problem and being willing to change the model completely were mentioned by intermediates as well, but unlike the experts these were outliers as opposed to major themes.

When novices were asked about the barriers they encountered in the MCM they mentioned several different issues, including working with others in a group (Section 6.6). Several methods were attempted to move from being stuck including the use of numerics and switching strategies. One novice mentioned changing modelling assumptions and and another mentioned collaboration as a means of getting unstuck. What is noticeable is that none of them spoke of overcoming the barriers successfully, nor did they discuss understanding why they were stuck. Students' responses about whether they found working in a group useful were split. Students often cited the group dynamic as being a part of the problem although they acknowledged that it can be useful as well. This is in contrast to the experts but not unexpected. These novice modellers have limited experience with working in groups, and therefore have not developed their mathematical or even professional collaborative skills, something deemed important by the experts (Section 4.5).

For the complete novices, as one might expect, there is a marked difference in how they deal with becoming unstuck. The first thing of note is that this group of participants bring forth the new theme of giving up completely. This is not mentioned by any of the experts, intermediates or MCM novices. Those complete novices who do not give up spoke of re-reading and re-doing the question, often without a discourse on trying to understand exactly why they were stuck. Another major theme raised by this group was waiting for the answer, as opposed to collaborating with their colleagues. This theme of learned helplessness [55] is unique in this study to the group of complete novices. Although other strategies were mentioned including breaking down the problem, drawing a picture, trying something and exploiting a similar problem, these were in the minority being mentioned by six or less students each out of a total of 45 students.

### 8.4.2 The Psychology of Modelling

We have seen above that the experts collaborate when stuck, while the intermediates turned to research. The novice modellers attempted several methods of getting unstuck that were for the most part unsuccessful, and the complete novices tended to give up, wait for help or re-do the problem, with no mention of changed assumptions. We know from the literature that experts have a broader knowledge and better cognitive and metacognitive skills, but what psychological factors affect expert and student success?

The first psychological factor explained by Andrea Bertozzi is that experts make the decision about what problems they work on, while students do not posses similar autonomy of choice. As David Muraki explains, there are expectations (by both the instructor and the student) that when given a problem, the student should be able to do it, whether or not the student feels capable enough or interested enough to do so (Sections 4.2, 4.3). Autonomy is one of the three main intrinsic motivators quoted by Pink [41] when discussing the work of Deci and Ryan: "Autonomous motivation involves behaving with a full sense of volition and choice" [43]. This lack of autonomy for the novices and to some extent for intermediates, implies a lack of intrinsic motivation as well, which is one of the main elements Type I behaviour and thus expertise (see Section 2.3.2).

The second psychological factor of successful modelling is confidence and risk-taking. Throughout the interview experts speak about being fearless, being willing to try something, and not worrying about whether the initial guess is wrong as they can learn from it. Mike explicitly states that one must be willing to go out on a limb and not worry about getting it wrong (Section 4.5). Sam Howison also speaks about not being afraid to ask questions (Section 4.7). This explains why collaboration when stuck is a big favourite among these experts. Fear of asking questions or fear of failing makes it difficult to collaborate and indeed to make any progress at all.

Tobias mentions fear of making mistakes as one of the reasons that students would experience anxiety and refuse to voice mathematical thoughts: "One thing that may contribute to a student's passivity is the fear of making mistakes in mathematics [...] Successful math students know better. They do not despise their errors."( [13], p.52). This attitude, prevalent among expert modellers, becomes less evident as we move down the spectrum. Thus we see experts willing to discuss their problems with anyone, intermediates discussing with colleagues or supervisors, novices having a hard time collaborating with each other, and acute novices simply giving up or waiting passively.

The experts in this study described interest and curiosity for the most part when faced with modelling problems, particularly describing difficult problems as being more interesting (Section 4.10). In cases where worry or anxiety was present, the experts were able to push past those feelings and make progress. When stuck, some experts spoke of waiting for insight, implying that they expected the ideas would be subconsciously solved. Experts described these difficult problems as the ones where they learned a lot. These qualities hint at creativity in the Hadamard sense as well as from Csikszentmihalyi's perspective. The experts are accustomed to illumination as being part of their process, which Hadamard's research also supports. The experts also do not crave the easy problem, but prefer the problem to be a bit out of their reach. This implies a search for flow where the problems are not too easy, which in turn lead to satisfaction and mastery [41].

The complete novice experiences frustration as opposed to motivation when stuck or if the problem is difficult (Chapter 7). There is evidence that the MCM novices feel some interest in the face of mathematics as they volunteered to take part in the competition. However their discussion about the barriers they encountered also suggests a great deal of frustration (Chapter 6). Intermediates on the other hand experience a mix of interest and worry when first faced with a modelling problem, often explaining that these feelings are problem dependent. Some intermediates described waiting for insight or illumination, pointing to creativity in the Hadamard sense, though when describing difficult problems there is no evidence that these are the problems they find more interesting (Section 5.10). The fact that intermediates speak to feeling overwhelmed (Section 5.2), suggests that they are not experiencing the enjoyment of flow, but rather suffer some anxiety.

These beliefs, attitudes and feelings towards mathematical modelling have a significant impact on motivation and confidence, which in turn impact mastery when modelling. This is particularly evident when stuck.

### 8.4.3 Defining Mathematical Modelling

The experts of this study define mathematical modelling in two ways (Section 4.1). The first is considering modelling as a simplified description of a real world situation understood using mathematical formulation. In the second definition modelling is seen as a process starting with the creation of a mathematical framework to describe a real world problem, followed by the solution and refinement of the mathematical problem and a return to the real world problem in order to explain or make predictions.

The intermediates did not have this dichotomy in their definition, describing mathematical modelling as breaking down a real-world problem and using mathematical language to explore it (Section 5.1). In this group the solution of the modelling problem is incorporated into the definition. None of these intermediates spoke of refining the model or using the model to make predictions. This may be indicative of the fact that they are more likely modelling toy problems than problems directly from industry.

For the novice MCM participants what makes a problem a modelling problem is even more simplistic. This group of modellers describe modelling as a mathematical representation of a real life problem. Again, no description of refining the model or predictions is evident. The dichotomy noticed here was that some students depicted a modelling problem as one that had more than one solution method. This is not a defining condition of modelling for the experts or in the literature, although it tends to be the case in reality.

The group of complete novices have not been introduced to the concept of modelling in their FAN X99 class, or possibly elsewhere. With this in mind this group was not asked to discuss their definition of a modelling problem.

We have examined the psychology of modelling for the different groups, particularly how each group deals with being stuck and defines modelling. We will now traverse the modelling process, taking a look at how each group starts a modelling problem, what skills they deem important, and finally how they verify their solution.

### 8.4.4 Starting a Modelling Problem

Many of the experts interviewed begin problems with excitement and curiosity, although they needed additional prompting to discuss their feelings. Some of them described feelings of fear or worry, but were able to push past those feelings of self-doubt (see Section 4.2). These experts begin by ensuring they understand the problem, which corroborates findings in the literature [18, 34]. This study reveals that the experts do this through research, exploration and most importantly collaboration. Their first plan is generally to begin with the simplest possible problem. Key to this approach is that it is acceptable if this simplest problem is not accurate, as it is more important that it captures some quality of the problem and gives some understanding of the overarching processes (Section 4.3). Experts focus on the big picture to determine relevance: this again is verified in the literature, which attributes this characteristic to a superior expertise [18]. What the literature does not tell us is that experts focus on the areas of the problem that can be best captured by the mathematical tools they are most familiar with (Section 4.4).

In comparison, the intermediates begin modelling with curiosity or persistent self-doubt, and were more willing to discuss their feelings than the experts (Section 5.2). Their initial strategy tends towards undirected independent research via the internet, textbooks or papers. Unlike the experts, no mention of collaboration with others was noted. There is a distinction here between what the experts and the intermediates view as research. For the intermediates, research is via literature and the written word. Alternatively for the experts, research implies seeing what others have done. This may require a look at the literature but can also be achieved by communicating with others.

Simplification was a heuristic mentioned, particularly by those intermediates who felt overwhelmed with the initial complexity of the problem. As with the experts the theme of trying to identify the dominant process of the problem was raised. Intermediates indicated that determining what was relevant to the model was difficult for them, either in the length of time it took them to respond, or by explicitly stating that this step was tricky (Section 5.4). Some of them attributed this skill to experience or intuition, without explaining further what this entails.

The novice modellers of the MCM exhibited a range of confidence with their modelling problem, from viewing it as “a piece of cake” to feeling “completely clueless” about how to go about solving it. The literature tells us that novices tend to be quick to start on a problem whereas experts take the time to understand and analyse the problem [35]. Strangely, starting immediately was only reported by three of the eight novice modellers. However, of the five that did not start immediately, three of them did not because the team was not assembled immediately, and one of them did not because he thought the problem was easy. Only one student mentioned trying to understand the problem first, and this was done via blind research (i.e. Google search). In determining what terms were important to the model, a variety of responses were noted, from everything was important to too many things were unimportant. For the most part these students did not give justification for why they thought a particular thing was unimportant to the model.

Approximately three quarters of the acute novices or FAN X99 students described negative emotions when first faced with a word problem (Chapter 7). These students tended to plunge into a solution as opposed to planning, with 28 plungers and 10 planners (Section 7.4). In direct contrast a majority described themselves as organised, with 28 claiming to be organised and 11 claiming to be disorganised (Section 7.5). This does not correlate with the literature [25] and may indicate that students are unaware of their own lack of organisation (*i.e.* the 'Dunning-Kruger effect' [51]), or that students consider their work organised due to neatness and handwriting skills, as opposed to organisation of thought. These students generally had a preconception that word problems should be solved in the order of minutes (Section 7.1). There was no discussion of determining what aspects were relevant to the model as the students had no experience with such problems.

### 8.4.5 Knowledge and Skills Valued

When I interviewed the experts and novices with respect to the skills they deemed important for mathematical modelling (question 5), I split the question into two parts. I identified that there were mathematics skills that were important, but I was also interested in non-mathematical skills deemed useful for modelling. This yielded a range of rich responses from experts and intermediates alike. Question 7 asked specifically for heuristics in the interview. This question was mirrored in the questionnaires given to both groups of novices. I will therefore address all three sets of skills in this discussion: mathematical skills, non-mathematical skills, and heuristics. Note that the complete novices were not asked what mathematics they used.

#### Mathematics

A look at the mathematical skills deemed important by the experts shows a wide variety of topics. As the group of modellers interviewed worked primarily with continuum modelling, it is not surprising that Calculus, ODEs and PDEs come out on top. Several experts also mentioned Statistics and Probability and Numerical Analysis (Section 4.5). Of course, answers were not limited to these 5 subjects and went on to include nine other topics: Linear Algebra, Abstract Algebra, Analysis, Data Analysis, Queueing Theory, Graph Theory, Discrete mathematics, Calculus of Variations and Optimisation. Many experts mentioned that the topics they listed were by no means exhaustive.

The majority of intermediates in this study also focused on continuum mechanics in their studies. This yielded Calculus and DEs as the more popular subjects mentioned (Section 5.5). This was followed by Discrete mathematics and Numerical Analysis. Other topics mentioned were not as varied as with the experts: Probability and Statistics, Linear

Algebra, Lagrangian Mechanics, Fourier Analysis and Analysis.

When asked what mathematics was used in the MCM, students mentioned several different areas, with a slight focus on Calculus and Physics. Other topics mentioned were varied: Euler-Lagrange Equations, Mechanics, Geometry, ODEs, Curve fitting, Discrete mathematics and "software." This is particularly interesting because all of these MCM participants worked on the same problem. This illustrates that one problem can appear to require many different areas of mathematics. I say appear because none of the students spoke of a particular area being more successful than another, and so we cannot conclude that any or all of the mathematics they used was indeed useful.

Experts report a wide range of mathematical topics as useful for a wide range of mathematical problems. Intermediates list a more restricted range of topics, which we expect from the literature [18]. Intermediates in the study tended to mention topics they were more familiar with (Section 5.5). When dealing with a single problem, the novices supply an equally varied list as the intermediates, although they make no claims that all of these methods were successful in finding a solution to the MCM problem. We also saw that the novices often were not familiar with topics suggested by their colleagues (Chapter 6). This suggests that the novices were not sure how to go about solving the problem and simply used all the knowledge available that seemed useful.

### Non-Mathematical skills

This ability to determine what mathematics to use is a metacognitive skill, as opposed to a purely cognitive one. This brings us to an examination of the differences from novice to expert in non-mathematical skills. Experts first discuss that breadth as opposed to specific mathematical skills are essential. This is evident in the breadth of mathematics topics that they were able to supply, but also in the knowledge that they brought from outside of mathematics. Not surprisingly we note that half of the experts actually speak about needing scientific knowledge related to the area that the problem is coming from.

Another very important skill is collaboration, a theme that has been raised throughout this thesis. Collaborative skills including communication, diplomacy, listening and patience with others were mentioned specifically by eight of the experts. Since this theme recurs throughout this study, it may be worthwhile to note here that there are several possible reasons why non-experts do not collaborate. In particular for the intermediates, as graduate students, needing to demonstrate independent thinking and research may cause them to shy away from collaborating too extensively (they may even be told explicitly not to collaborate by their advisors). We therefore distinguish between active collaboration (as seen by the experts) and passively asking for help (as exhibited by some complete novices). There is also a distinction within the different ways experts use collaboration. Collaboration is used to help clarify understanding to allow, for example, the formulation phase of modelling. Collaboration is also used as a means to achieve metacognitive goals *i.e.* to articulate one's thinking allowing one to pinpoint exactly where the difficulty is and thus move from a state of being stuck to becoming unstuck.

Apart from these three major themes, one expert also mentioned being able to organise one's thoughts, which ties in with the ability to determine what mathematics to use. This metacognitive skill is one which the novices lacked and is also supported by the literature [18] on the difference between novices and experts. Two experts highlight the need to understand what the words mean, as people of different disciplines use different terminology to describe similar topics. The experts mentioned several other personality traits as being invaluable: maturity, confidence, passion, curiosity, flexibility, stamina, persistence, hard work and patience. These characteristics are all essential aspects for motivation and creativity [41,45].

The intermediates interviewed mentioned several of the themes raised by the experts. Intermediates discussed themes of breadth and relevant skills, as well as scientific or background knowledge. Four intermediates raised the theme of collaboration, which was seen as a major skill-set by the experts, however this was approached in a slightly different manner. The experts here listed different aspects of collaboration that were useful, while intermediates spoke of a willingness to collaborate and not being afraid to ask questions (Section 5.5). Intermediates also tended to use collaboration in a metacognitive way (to pinpoint the difficulty when stuck) as opposed to using it as a means of clarifying the modelling problem in the initial stages (opting to turn to the literature instead). Intermediates are also inclined to talk to mentors, which in many cases is not an available option for experts. One intermediate (who was also a PDF) raised the theme of knowing what the words mean. This comes from work in the field with people of different disciplines who describe the same concept differently. This must be highlighted in the classroom, so that students know to first clarify the meanings of terms being used.

One other theme evidenced by the intermediates was the metacognitive skill mentioned earlier of knowing how to approach problems. Research was the number one tool for getting unstuck among intermediates so there is no surprise that being able to search for what you don't know was mentioned as a useful skill. Intermediates mentioned several personality traits raised by the experts, but in each case the context or wording is slightly different. Intermediates spoke of experience as opposed to maturity, and while passion, interest, curiosity, perseverance and hard work were mentioned they were not major themes as with the experts. Confidence was mentioned, but half of the intermediates that spoke of it stated it from a negative perspective, such as 'not being afraid of failure', as opposed to 'you need confidence.'

There were no questions on the questionnaire for the group of novice modellers that specifically addressed the non-mathematical aspects of modelling. However, a look at their answers to several different questions gave evidence to their deficiency in the skills valued by experts and intermediates alike.

These novice students valued collaboration, but often described it as not being helpful (Section 6.6), and instead listed it as one of the barriers to finding a solution to the problem. Similarly, they spoke of their lack of breadth of knowledge in response to barriers that were encountered by the groups. The metacognitive skill of knowing what mathematics to use was lacking, evidenced by the range of mathematics employed to solve the problem. One novice explained, "it was hard to determine what the question was asking" (Section 6.7), which also shows that understanding the problem was an issue. On the other hand novices exhibited interest by volunteering for the modelling competition. There was also evidence of confidence as they were willing to try some strategies despite not knowing if they would work. In some cases this confidence was misplaced (Section 6.7).

The complete novices were also not asked specific questions about what skills they deemed important for modelling, particularly because they have no real concept of modelling. Still, a look at their responses indicates several non-mathematical issues that they have that would hinder their modelling ability. Although some students spoke of a transition to more positive feelings as the semester progressed, for the most part feelings described were negative. Also noted is a tendency to give up or wait for help instead of actively participating, in contrast to the perseverance and hard work mentioned by the experts. Finally, these students tend to plunge rather than plan their solution, while experts recommend patience and ensuring you understand the problem first.

**Heuristics and Problem Solving Strategies**

The experts interviewed in this study cited five main heuristics (Section 4.7). The first of these is *drawing a picture*. This heuristic was mentioned as a means to clarify understanding, in order to help those working on the problem or those bringing the problem to see what is happening.

This ties in with the second heuristic: *understanding the problem*. This supports Polya's framework, where understanding the problem is the first step. This includes an understanding of what the words being used mean, which was mentioned as an important non-mathematical skill. A look at non-mathematical skills of novice modellers show that understanding the problem is a heuristic that is lacking.

The third heuristic of the experts is *simplifying the problem*, which is also connected to understanding the problem. This heuristic is mentioned in the literature [18], but two experts clarify further that this entails looking at limiting cases in particular as a means of simplification in applied mathematics. For the experts looking at limiting cases gives an insight into the problem and yields a better understanding of the more complex cases. Limiting cases would fall under the category of special cases in Briggs' work [16].

The fourth heuristic the experts mention is *exploiting a related problem*, which is also mentioned by Briggs. This requires experience, a skill broached by the intermediates; and the ability to recognise which problems are related, which is a metacognitive skill. Experts describe an easy problem as one that is familiar, which encompasses this heuristic of exploiting a related problem, as well as the ability to recognise which problems are related. Being able to reformulate a problem into one you know so that you can exploit the related problem is a high level skill.

The final heuristic mentioned by the experts is *talking to others*. This is no surprise as experts throughout this work have mentioned the themes of collaboration and communication. This is not seen in any of the literature on modelling; however, this is observed in the mathematical modelling camps and workshops. The breadth of knowledge required to solve an industrial modelling problem makes it necessary for a collaborative effort. As one expert succinctly put it, "mathematical modelling is not a solitary activity" (Bob).

Having looked at the heuristics of the experts we now turn to those employed by the intermediate modellers. Recall that for six of the eleven intermediates, a full list of possible heuristics were provided, making the responses somewhat skewed. I will therefore only discuss the main heuristics noted.

The first two main heuristics mentioned by the intermediates are *drawing a picture*, and *solving a simpler problem*, with nine intermediates each mentioning these. These two heuristics were similarly discussed by the experts. Three of the eleven intermediates also mentioned the heuristic of *collaboration*. While not nearly as prominent a heuristic as described by the experts, it is evident nonetheless.

Intermediates also mentioned a fourth heuristic of *taking a break*, which was not suggested by experts in response to this question (Sections 4.7, 5.7). This heuristic is of note as not only did all intermediates prompted agree, but three others not prompted also suggested that taking a break was a useful heuristic.

The novice participants of the MCM, mentioned only two heuristics, when asked what strategies they tried and why. The first heuristic mentioned was *research*. This was evidenced in responses by the intermediates but was not a heuristic included by the expert. The second heuristic was *simplification*. This was stated specifically by one MCM participant, and was also implied by another participant who started with the actual shape of a snowboard before trying a more standard shape. Novices also mentioned in their problem-solving strategies: use of 2D/3D surfaces, complex physics and complex numerics, continuous and discrete mathematics, Newtonian mechanics and Lagrangian mechanics. These responses suggest that the mathematics used and the problem-solving strategies employed were not distinct for the novices.

A look at the problem-solving strategies of the complete novices reveals a similar finding. Students mention mathematics skills such as algebra and mental math as their problem-solving strategy. The main heuristic seen here was *drawing a picture*, with an overwhelming number of 36 out of 45 students claiming to use this strategy. Students also mentioned *working backwards, exploiting a similar or simpler problem, relating the problem to real life* and *using logic*. Working backwards is not noted as a useful skill in modelling by either the intermediates or the experts, but these students experienced some word problems for which this made sense. Relating the problem to real life is a very useful heuristic to transfer to modelling, as this is the very essence of modelling. While thinking logically is used by all groups, it is mentioned by none of the others because for every group except the complete novices, it goes without saying.

### 8.4.6 Verifying the Solution

We now look at the final stage of the modelling process – verifying the solution. The experts immediately clarified that in modelling there is no correct solution: the end of the modelling process is not checking that you are right, but that your solution is sensible. Experts also comment that verification of the solution is necessary. To do this experts suggest first and foremost a comparison with the data or experimental results. Arising from this theme is the fact that one has to be careful that the data used to create the model is not the same as the data used to test it. Other experts speak of making predictions, which would also require a comparison with data to test if the predictions are meaningful. Looking at limiting cases is mentioned again, here as a means of verifying the viability of the model or solution. Experts suggest that you will know your solution is sensible by looking at the qualitative behaviour of the model. Two experts also suggest comparing two different solution methods, especially if no data was available for testing.

Intermediates were not given an opportunity to explain that there is no correct solution to a modelling problem as I had already changed this question in the interview. Intermediates mentioned themes raised by the experts including: comparing with the data, that the model makes sense, looking at simple test cases, and comparing two solutions.

The intermediate participants also raised themes not mentioned by the experts. The first of these is that intuition will let you know if the solution is sensible or not. Only one expert speaks to intuition, however the intermediates raised this theme on more than one occasion. Intuition is defined as the ability to understand something immediately without need for conscious thought. This ties in with the Dreyfus model of expertise when pertaining to experts, who find it hard to explain their processes as they have become automatic. This suggests that the experts do not mention intuition because the process is too embedded in their subconscious to make it obvious for discussion. An intermediate looking on will observe this as an expert seemingly arriving at the solution without thinking.

The second theme raised by the intermediates not seen in the responses from the experts is convergence testing. Two intermediates describe convergence testing as a means of checking if their solution is sensible. Convergence testing will verify if the numerical scheme is working to the order it should and if the results are converging as expected, but does not show if the solution is applicable to the real world problem itself. However, this is a viable way to test the numerics of the problem. This concentration on checking the numerics is not an expert behaviour that is noted in the literature or from this study. It is perhaps evidence of the intermediates losing sight of the important goals and focusing on what they know and can do best. Testing the numerical models is an important part of scientific computing, but it neglects the fact that perfect algorithms give no insight when applied to incorrect models.

The complete novices offered several ways of checking if their solution was correct. Notice that there is no dispute about the correct solution as these students are used to one closed-form solution being the answer. Strategies for verifying the solution include checking that the answer makes sense by re-substituting, working backwards, and re-calculating. These methods have the potential to verify arithmetic errors, but do not give feedback about whether the initial equations (or model) are correct, or if those solutions answer the original word problem. Another method mentioned is checking with the instructor. This ties in with the novice's tendency to be passive and wait for help. This also speaks of a belief that others hold the keys to answers that students are not privy to. Finally, some complete novices admit that they do not check their solution at all, in contrast to experts who deem this to be a necessary step.

The novices that competed in the MCM were not asked about verifying their results. However, we can infer from the numerous references to time constraints that verification may not have taken place.

## 8.5 Summary

These conclusions have implications for the teaching of mathematical modelling, if the aim is to help develop the novice modeller become an expert. While the established literature addresses the cognitive and metacognitive differences between the novice and the expert, there are still several things lacking. These include a definition of modelling, an investigation of how one moves from being stuck to becoming unstuck, and a discussion of the non-cognitive differences between the expert and the non-expert. This study has addressed these three issues in particular.

We have seen in the modelling literature that there is no agreed upon definition of modelling. Among the participants of this study there are also differences. The experts exhibit a dichotomy in their responses, with some viewing modelling as the formulation of the model and others viewing modelling as the entire process including verifying and refining the model. The intermediates focus on the solution step of the modelling process but expressed a similar definition to that of the experts. On the other hand, some novices misunderstood what modelling is, assuming that modelling problems are ambiguous by definition, as opposed to being ambiguous as a consequence of coming from real-world problems.

When dealing with being stuck, the experts tend to collaborate with others around them, those who have brought the problem, colleagues, and even those who have not worked on the problem at all. This is not seen in the modelling literature but was raised by almost every expert interviewed. Most important for the experts is understanding the problem in order to become unstuck. The intermediates turned primarily to the literature to increase understanding as opposed to collaboration. The novices spoke of switching strategies when stuck without discussing trying to understand why they are stuck. Complete novices were the only group to mention giving up completely when stuck. They also tended towards more passively asking for help or waiting as opposed to active collaboration.

There are several other differences as we traverse the landscape from novice to expert. There is an increase in autonomy as we move along the spectrum, with the complete novices having little or no autonomy and the experts having almost complete autonomy. There is also a decrease in persistent self-doubt or anxiety as we travel along the spectrum from novice to expert. Experts do speak of feeling some anxiety, although they are able to distance themselves from these emotions in order to address the modelling problem. Experts also described difficult problems as interesting, while complete novices saw them as frustrating.

These results clearly indicate that simply teaching more mathematics is not enough, especially as the literature and the findings of this study suggest that psychological, as well as metacognitive characteristics need to be developed alongside the cognitive in order to aid with success in modelling. We will discuss these implications for teaching as well as future work in the final chapter.

# Chapter 9

# A Look to the Future

There is still much work to be done to fully understand all of the aspects of the modelling process. This qualitative study has hinted at several of the different issues that need to be dealt with when modelling and more work can be done to tease these out further. Having started out with the aim of understanding how to learn modelling, and what the learning outcomes of a modelling class would be, it is only logical that I look at the implications this work has for the teaching of mathematical modelling. Therefore in this chapter I will look at the implications for the teaching of modelling, followed by suggestions of what future work must still be done.

## 9.1 Implications for Teaching

From the literature on cognitive and meta-cognitive processes [12, 18, 19], we know that these as well as maturity and experience take time to develop. These are not skills that can be taught, but are a result of motivated work in a particular area. However, from this work there are definite skills that can be taught to help modellers on their path to mastery.

The first implication of this work is that a working definition should be given to students. While it is acknowledged that modelling is a vast topic, there are still some key features of modelling that experts and literature agree upon. A discussion in the classroom of the dichotomy between modelling as creating the model, and modelling as creating and solving the model, should give students a clear idea of what the process entails. This would help them to identify when they are faced with a modelling problem. It also aids in developing learning outcomes: which aspect of the modelling process is the class focusing on, the creation of models, the solution of models or both?

From the literature on problem solving and the responses of the experts, a focus on trying to first understand the problem should be a major part of the modelling classroom. The literature on novices as well as evidence of this work indicates a tendency of novices to plunge in without first understanding what is happening. Experts on the other hand spend a large proportion of the time trying to fully understand the problem including: understanding what the words mean, understanding exactly what is being asked, understanding what aspects are relevant by looking for dominant processes, and obtaining background knowledge in the field the problem originates from. Classroom practices can mirror this, taking the focus off trying to solve the problem initially, and just trying to understand the problem. Explicitly stating that understanding the problem is the objective, helps to re-direct many students' unspoken belief that the point of mathematics is getting a quick solution as opposed to a verifiable one.

A third implication of the study is that students in the study were unable to separate the mathematics from the heuristic. Why is this important? If the point of using a certain area in mathematics is to help simplify a particular problem, then students may tend to use that area in mathematics at all times, whether it causes simplification or not. Separating the mathematics from the underlying principle allows them to access a wider range of mathematics in theory.

Another deficit identified from the study is the non-experts' inability to verify solutions, focusing instead on checking numerics, checking arithmetic, or simply skipping this step. This step should not be overlooked or glossed over in the classroom, as it is a key step in the modelling process and deemed essential by the experts. A focus on verifying the solution needs to be evident in the classroom if we want students in turn to focus on this aspect when completing the modelling process. This means including the refinement process as well as comparison to data, and a discussion of how to verify a solution by looking at qualitative behaviour when no data is available.

A major focus of this work was helping students move from a state of being stuck to becoming unstuck. So far all of the areas discussed are directly derived from the modelling process and may be more obvious to a teacher of modelling. However helping students move from stuck to unstuck is more subtle. Since collaboration and communication are the main means of doing this by the experts then this must be encouraged. Students must also be encouraged to focus on why they are stuck. Intermediate responses are helpful here, as they suggest that trying to articulate why you are stuck often helps the transition to becoming unstuck. This is also seen in the problem-solving literature. This again leads us back to collaboration and communication. In applied mathematics modelling classes a focus on the use of limiting cases in an attempt to simplify is also useful. Students would also benefit from seeing real modelling done by experts. Seeing experts deal with being stuck is informative, and helps change the belief that experts simply rely on intuition. Finally, participating in modelling camps, competitions and workshops should be encouraged, as much of modelling is learnt from actually experiencing modelling in this collaborative environment, where students can see first-hand how to use collaboration as a means of getting unstuck.

In the previous paragraph we saw the theme of collaboration emerge as a means of helping students and experts alike move from stuck to unstuck. However, collaborative skills are not automatic. This is especially true in mathematics classrooms where focus on individual skills is usually the norm. Thus, collaborative skills must be developed. This means that modelling classes cannot be solely lecture based to be efficient. Students need to work on the ability to articulate their thoughts, the ability to be diplomatic, the ability to listen respectfully and to have patience with each other's ideas. The environment must also be one in which students are willing to collaborate and not afraid to ask questions. Discussion with the mentor or teacher should be encouraged not only as a means of clarifying understanding, but also to help students break away from passivity.

The final implication for the teaching of modelling that we shall discuss here is that of autonomy. Experts choose the general area they work in, as well as the specific problems that they want to work on. This autonomy leads to intrinsic motivation as they are invested in understanding the problem and coming to a viable solution. Similarly, in modelling camps student have some choice of whether to attend, and which problem to work on. In order to help facilitate this intrinsic motivation, teachers must provide problems that mathematics students are familiar and confident with, and give the option to work on different problems if one is not of interest. Pink [41] explains that for non-routine problems intrinsic motivation is vital. Mathematical modelling is not routine due to the breadth and variety of problems it covers and the fact that those problems come from the real world and are messy. This therefore implies that mathematical modelling cannot be mastered without self-motivation and interest.

## 9.2 Future Work

The work that is discussed in this thesis is only a small portion of what needs to be done to fully understand the modelling process (if it can ever be fully understood). In an effort to get a clearer view of the issues several extensions of this study can be explored. The first of these is a more varied participant group. While the experts and intermediates came from different backgrounds, they were for the most part from North America and Western Europe. A look at how modelling is viewed across the world may give rise to an even wider variation in definitions, skills, and beliefs of modelling. The novices in the study all came from SFU and it would be interesting to see if these traits hold true across universities. I would also be interested in noting the differences between novice modellers in different subject areas.

I chose 4 broad categories for the purpose of this study, due to the ability to easily identify the two extremes. The intermediate group represented all persons that could not fall under the expert or novice categories. However, the Dreyfus model describes five different stages from novice to expert. It would be interesting to try to classify the intermediate participants based on the skills that they have for a more specific look at a range of modelling ability. It would be especially interesting to see if the level of expertise automatically increases based on the number of years spent modelling, and if not, why some intermediates are more advanced than others with fewer years of experience. The goal orientation of the modeller also appears to change with expertise (experts tend to be motivated to learn while novices tend towards avoidance). This suggests the development of tools to assess the goal orientation of modellers from particular groups.

In this study I made use of observations to provide a context for understanding the modelling process, and being able to interpret the responses to interviews and questionnaires. However using those observations to verify or contradict responses would reveal an even deeper understanding of the modelling process for all three groups. In particular it would be interesting to see how the novices' responses match to their actual process, if it were captured on video. I suspect that for the intermediate group this would be a challenge, as at the intermediate stage there is more awareness of what is unknown than at the novice stage, but not as much confidence as at the expert stage. Thus I believe that intermediates would have the least desire to have their actual process recorded.

Of course, one of the obvious directions for future work is the implementation of the suggestions in relation to teaching, to see if they lead to a more successful modelling experience for students. In order to keep track of the affective and extra-logical aspects of modelling we could have students keep a modelling journal, describing their understanding of the process, their struggles, and their triumphs. Journaling is not straightforward for mathematics students and may require some illustration [33] but the combination of the journals with classwork would give a clear picture of the logical, extra-logical, and psychological advances students are making as they travel along the spectrum towards modelling expertise.

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