THE AHA! EXPERIENCE: MATHEMATICAL CONTEXTS, PEDAGOGICAL IMPLICATIONS

by

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ABSTRACT

The AHA! experience is a term that captures the essence of the experience of illumination. In the context of 'doing' mathematics it is the *EXPERIENCE* of having an idea come to mind with brevity, suddenness, and a sense certainty. In the studies presented here I examine this extra-logical process in pursuit of the answers to three questions: *What is the essence of the AHA! experience? What is the effect of an AHA! experience on a learner? Can the AHA! experience be controlled, and if so can it be invoked?*

The data for this pursuit comes from three distinct sources; the anecdotal reflections of 76 undergraduate students, the anecdotal reflections of 25 prominent mathematicians, and the mathematics journals of 72 preservice teachers. The results indicate that, although the AHA! experience is precipitated by the sudden coming to mind of an idea, what actually sets the AHA! experience apart from other mathematical experiences is the affective components of the experience, and only the affective component. That is, what serves to make the experiences extraordinary is the affective response invoked by the experience of an untimely and unanticipated presentation of an idea or solution, not the mystery of the process, and not the idea itself. Hence, the AHA! experience has a positive and, sometimes profound, transformative effect on a learner's beliefs and attitudes about mathematics as well as their beliefs and attitudes about their ability to do mathematics. The results also indicate that a measure of control can be exercised over the AHA! experience through the manipulation of a problem solving environment. As such, the results provide a pedagogical approach to problem solving that can be used in the classroom. Furthermore, a methodological contribution in the form of a new form of journaling for the tracking of students' problem solving processes is presented.

For my wife, Theresa.

Once again you have found it in you to give me the time and the space to pursue my dreams, my passions, and my musings.

For my children, Anders, Connor, and Lena.

Thank you for your patience, your love, and your unfailing smiles.

QUOTATIONS

Disgusted at my want of success, I went away to spend a few days at the seaside, and thought of entirely different things. One day, as I was walking on the cliff, the idea came to me, again with the same characteristics of brevity, suddenness, and immediate certainty …

- Henri Poincaré (1952, pp. 53-54)

My attitude towards mathematics is that most of it is lying out there, sometimes in hidden places, like gems encased in a rock. You don't see them on the surface, but you sense that they must be there and you try to imagine where they are hidden. Suddenly, they gleam brightly in your face and you don't know how you stumbled upon them. Maybe they always were in plain view, and we all are blind from time to time.

- Enrico Bombieri, mathematician[1]

I must admit that math is challenging for me … after the AHA! experience you feel like learning more, because the joy of obtaining the answer is so exhilarating. It almost refreshes one's mind and makes them want to persist and discover more answers. It gave me the inspiration and the determination to do the best that I can do in the subject.

- Kristie, mathematics student[2]

[1] Excerpt from data presented in chapter six.
[2] Excerpt from data presented in chapter five.

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FOREWORD

I have always enjoyed mathematics, and I suppose it would be fitting for me to say that the reason for my enjoyment of the subject is due to the many AHA! experiences that I've been privileged with in my mathematical encounters. In part, at least, I would say that this is true. But to say that all of my passion for mathematics comes from such experiences would be false. I think that for me, as for many others who enjoy this subject from an early age, mathematics is a place where I can engage in a type of thought play that isn't possible in any other context. Certainly, discoveries and epiphanies have an important role in such play, but I simply enjoy the game itself.

Having said that, however, I have always taken particular pleasure in AHA! experiences. They make up my most memorable moments in mathematics, and they call to me, and keep me looking in anticipation of the next one. My first such experience was at the age of six when I figured out that multiplication – which I had heard mention of – was really just repeated addition and, as such, something that I could do. In the grand scheme of things, this is no great revelation, but for me it was monumental. I had grasped the essence of multiplication, and I had done so in a flash of insight. I was excited. My mind was racing to test my discovery. I remember frantically trying to explain to my father what it was that I had found. Whether or not he was impressed, I don't recall, but I clearly remember my desire to share my findings immediately. As the moment faded I was left with a residual satisfaction, a feeling of accomplishment, and an augmented sense of confidence.

As I got older and pursued mathematics more vigorously these AHA! experiences became an anticipated and accepted part of doing mathematics. I never took them for granted, but somehow I lost sight of the fact that these were very special moments, and as much as I anticipated and accepted them, they were still quite rare. As I amassed more and more such experiences the wonder of the AHA! began to wane. The residual satisfaction that the AHA!'s left behind never diminished, but it took on less prominence. It wasn't until I entered the realm of mathematics education research that the significance of these rare moments returned. I was reawakened to the wonder of the AHA! experience and chose to make it the focus of my research and my dissertation topic.

This reawakening didn't happen overnight, however. It took over a year, and involved three, seemingly independent and very different events. The first of these was in the form of a very powerful and memorable AHA! experience. I was enrolled in *Math 604: Geometry* (the third of six courses required for completion of a master's degree in mathematics education at SFU) taught by Rina Zazkis. One evening, as the class was drawing to a close, Rina posed a problem to us.

> *You're grandfather is ill and you are going to his house to visit him. On you're way you will stop at a nearby stream to collect some fresh water for him to drink. Both your house and your grandfather's house are on the same side of the stream, although not necessarily equidistant from it (see figure 1). At what point of the stream should you stop so that your journey is as short as possible?*

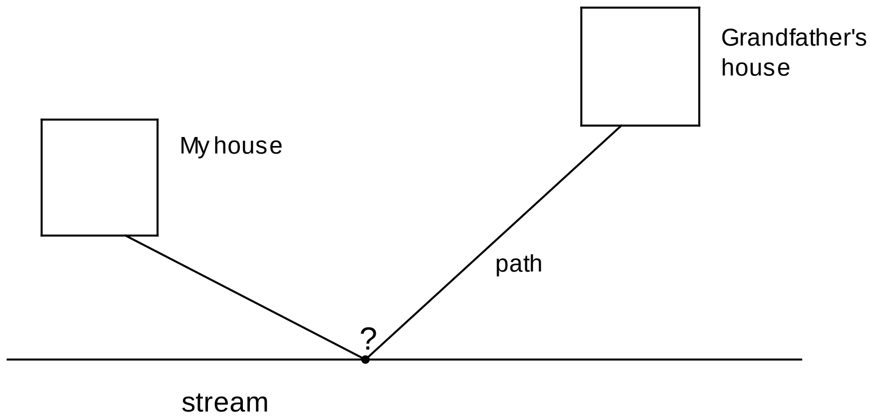

Figure 1: Path to Grandfather's House

There was still 30 minutes left in class so a classmate and I teamed up and attacked the problem. Over the next half hour we levied all of our calculus skills at this optimization problem. The problem did not yield. No solution emerged out of the mess of equations that we were generating. As the class drew to a close we still had no solution and Rina stated that she wanted us to work on it for the following week. As it was, however, I neglected, and eventually forgot, to work on the problem. When Rina began our next class with a request for a solution my mind was turned back to the problem for the first time in a week. In a *flash* the answer came to me. It was all right there; the solution was fully formed and in no way resembled my prior work on the problem. I saw now that the shortest path is achieved if you walk to the stream as if you're grandfather's house is in the same place, but on the other side of the stream (see figure 2).

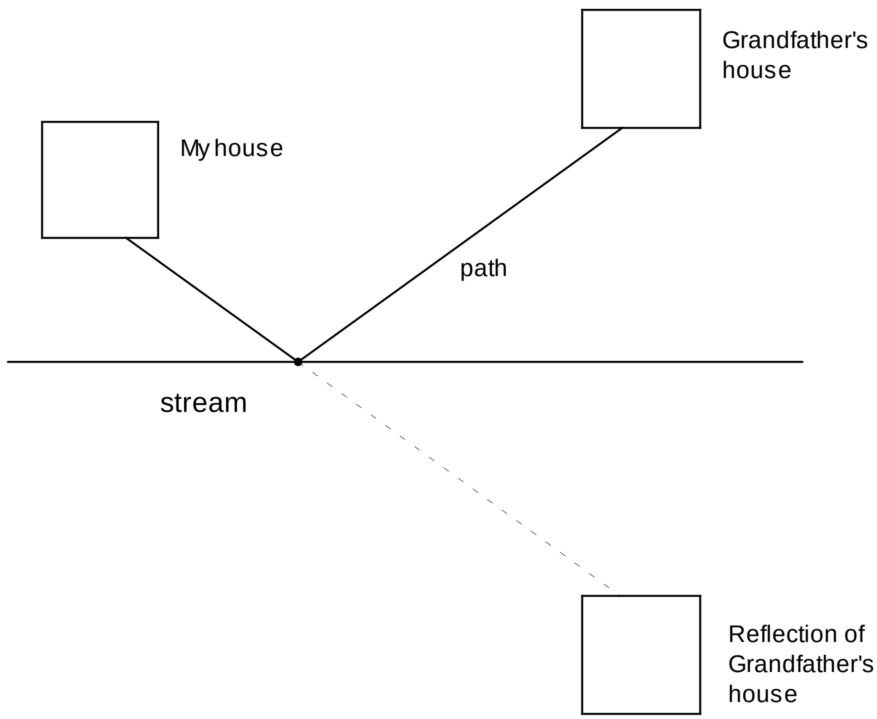

Figure 2: Shortest Path to Grandfather's House

I was, once again, eager to share my discovery and jumped at the chance to present my solution to the class. This AHA! experiences was different. It was *very intense*. The solution came to me fully formed, seemingly, from nowhere. I had not worked on it at all during the week, and my last efforts on the problem had been in a completely different direction. I was overwhelmed with a feeling of certainty; without conscious verification of the accuracy of the solution I knew it was correct.

The second event that moved me in the direction of pursuing AHA! experiences as my doctoral dissertation topic occurred almost exactly one year later. It had already been decided, and subsequently arranged, that I would abandon my pursuit of a master's degree and proceed directly into the PhD program. However, it had also been recommended to me, that rather than leave the master's program immediately, I complete the final course, *Educ 847: Teaching and Learning Mathematics*. This recommendation was made because the course was to be taught by two visiting instructors, David Pimm, and Bill Higginson, and an opportunity to be under their tutelage was not to be missed.

The first half of the course was taught by David, who very early on gave the class an article to read by Mary Barnes (2000) called *‘Magical’ moments in mathematics: Insights into the process of coming to know*. The article referred to these AHA! experiences as 'magical moments' in mathematics and dealt with the phenomenon from the perspective of teaching and learning mathematics, and of the effects that they had on students in a mathematics classroom. As a high school teacher I had always tried to get my students to have their own little epiphanies, to get 'the light to go on'. However, I had done so, not for them to experience the AHA!, but rather in order for them to 'get' the mathematics. I was leading them to their own discoveries in order that they achieve the end product, the understanding. Barnes' article, however, allowed me to see the AHA! as having the potential to do more than this. The AHA!! could improve a student's experience of mathematics and not just their understanding of mathematics. The AHA! could be used a pedagogical tool, as well as a didactical one.

It wasn't until September of that year, however, that all the pieces fell into place. I was in the process of clearing out some papers and reorganizing my file cabinet when I came across some literature on a SSHRC (Social Science and Humanities Research Council) fellowship. The application for this fellowship would require me to identify my research interest, do enough work in the area to show that I had mastery of the topic, and write a convincing summary of this work, as well as my proposed research plans. This was somewhat daunting considering that I had, to this point, been officially enrolled in the PhD program for only two weeks. As I pondered this prospect my eyes fell on the Barnes article that was lying on top of the mess of papers covering my desk. *In a flash* (ironically) I knew what my dissertation topic was to be. I was going to unlock the secret of the AHA! experience and harness it for the purposes of enhancing students' mathematical experiences.

And so began the journey. Along the way there have been many distractions, many digressions, and many pursuits of varying fruitfulness. I have come far and strayed further. I have ended up in a place very different from where I started and even more different from where I thought I would end up. My dissertation, in an attempt to make it a focused piece of writing, tells the story of the more fruitful explorations that I pursued, and the ones that most directly led to my conclusions. However, what is missing is no less important. It was through the early goings, through the many false leads and wrong turns that I learned the most. It was here that I first saw how elusive the AHA! was, and here that I first conceived of how best to attack it. At first I was naïve, idealistic, and wrong. But eventually I became smarter, more tactful, and more pragmatic about how to deal with this very intangible topic. Ideally I would like to include all this in my dissertation, but the sheer volume of it, would not only convolute my conclusions, it would serve to delineate my dissertation beyond hope of being thought of as anything that resembles an argument. As such, I have purged from the body of the dissertation the text that details these early explorations. What remains are the lessons learned and the conclusions arrived at.

INTRODUCTION

> *Perhaps I could best describe my experience of doing mathematics in terms of entering a dark mansion. One goes into the first room, and it's dark, completely dark. One stumbles around bumping into the furniture, and gradually, you learn where each piece of furniture is, and finally, after six months or so, you find the light switch. You turn it on, and suddenly, it's all illuminated.*
>
> \- Andrew Wiles[3]

Suddenly, it's all illuminated. In the time it takes to turn on a light the answer appears and all that came before it makes sense. A problem has just been solved, or a new piece of mathematics has been found, and it has happened in a flash of insight, in a moment of illumination, in an AHA! experience. From Archimedes to Andrew Wiles, from mathematicians to mathematics students, the AHA! experience is an elusive, yet real, part of 'doing' mathematics. Although it defies logic and resists explanation, it requires neither logic nor explanation to define it. The AHA! experience is self-defining. At the moment of insight, in the flash of understanding when everything seems to make sense and the answer is laid bare before you, you know it, and you call out – AHA!, I GOT IT! However, the AHA! experience is more than just this moment of insight. It is this moment of insight on the heels of lengthy, and seemingly fruitless, intentional effort. It is the turning on the light after six months of groping in the dark.

[3] From the movie 'The Proof', produced by Nova and aired on PBS on October 28, 1997 (Nova, 2003).

Literature is rich with examples of these AHA! experiences, from Amadeus Mozart's seemingly effortless compositions (Hadamard, 1945) to Samuel Taylor Coleridge's dream of Kubla Kahn (Ghiselin, 1952), from Leonardo da Vinci's ideas on flight (Perkins, 2000) to Albert Einstein's vision of riding a beam of light (Ghiselin, 1952), all of which exemplify the role of this elusive mental process in the advancement of human endeavours. In particular, scientific advancements are often associated with these flashes of insight, bringing forth new understandings and new theories in the blink of an eye. On a larger scale, the advancement of science in general serves as a nice metaphor for the AHA! experience. Over long periods of time science seems to progress at a steady rate, albeit an exponential one. Upon closer examination, however, what is revealed is a field that moves along in fits and spurts, with long periods of much activity and little progress punctuated with occasional flurries of advancement (Gardner, 1978). Even nature seems to take its cue from this phenomenon. Evolution, at first thought to be a slow and steady progression over time, is now being seen as lengthy periods of inactivity with occasional bursts of reorganization (Johnson, 2001). The AHA! experience is everywhere, from human endeavours to nature itself, and it punctuates the knowledge of the world around us.

As natural a part of our thinking processes as the AHA! experience is, for the most part it remains a mystery. Somewhere within the far reaches of our unconscious some mechanism is at work which allows our mind to produce these flashes of illumination, of this there is no doubt. Yet, these mechanisms remain hidden from us, and they may always do so. This does not prevent us from theorizing about its workings, however. From Gall's suggestion that mathematical creativity resides in a bump at the back of the scull (Hadamard, 1945), to Thom's catastrophe theory (Saunders, 1980), there is no shortage of hypothesis as to where this sudden appearance of an idea comes from. In the midst of all these theories, however, there are very few definitive answers. What is actually known about the AHA! experience can be distilled down to one concise statement: illumination occurs after a long period of conscious effort followed by a period of unconscious work (Hadamard, 1945; Poincaré, 1952). All else stems from this one understanding.

We know, for example, that the sudden appearance of an idea may occur to us in our sleep, upon waking, during conversation, or in the bath (Hadamard, 1945), but this is only a refinement of the understanding stated above with regards to where and when illumination occurs. We know that it is important to think hard and to think broadly about something if we wish for an AHA! to occur (Ghiselin, 1952; Perkins, 2000; Root-Bernstein & Root-Bernstein, 1999), but this is only a refinement with regards to the nature of the conscious work. We know that not all ideas that come to us in the flash of illumination are correct (Hadamard, 1945), but this is only a refinement with regards to the nature of the products of illumination. Clearly there is room for more to be known.

For the last three years I have been working on filling some of this room, to learn things about the AHA! experience that are not yet known. To do this I have restricted my efforts to the general context of mathematics and the specific context of mathematical problem solving. In particular, I have been interested in the distilling out the very *essence* of the AHA! experience within this context to determine what, if anything, about these experiences is common, and what sets them apart from other mathematical experiences. In the chapters that follow I present the details of a series of studies I engaged in to distil out this essence as well as to answer two further questions pertaining to the AHA! experience.

In the first chapter I discuss the AHA! experience in the context of existing literature on mathematical invention and discovery, as put forth by two great French mathematicians, Henri Poincaré and Jacques Hadamard. Their work is pivotal in our understanding of what it means to invent and discover in the field of mathematics, and as such formulates the underpinnings of any and all discussions on these topics. It was Poincaré who brought into the literature the very succinct idea that conscious effort is followed by unconscious work, and it was Hadamard that provided the empirical work to confirm it. This is followed by a detailed look at mathematical problem solving and the role that the AHA! experience plays in this most fundamental of all mathematical activities. Problem solving is what 'doing' mathematics is all about. Whether it be a student doing his homework, or a mathematician trying to forge new mathematics, they are both trying to solve a problem. As such, problem solving has come to play a very central role in the teaching and learning of mathematics. Problems are not just used as the context in which to engage in mathematical activity, they are also dealt with as a branch of mathematics, the skill of which is explicitly taught in the form of problem solving heuristics. In chapter two I examine how the AHA! is honoured and incorporated (or not) into a variety of different heuristics, from the seminal and much celebrated work of George Pólya (1957) to the controversial and intangible ideas of Gestalt psychology (Schoenfeld, 1985).

In chapter three I present the research questions I am interested in answering. This is followed in chapter four by a description of the general research methodology that I followed in the pursuit of these research questions. In this chapter I introduce each of three separate studies, the details of which comprise the next three chapters of the dissertation.

The first of these studies is presented in chapter five. This study was specifically designed to examine the effect of an AHA! experience on students who are resistant to mathematics and who have a phobia of mathematical content as well as of the learning of mathematical content. Although this study was initiated with the hope of enhancing an understanding of the cognitive aspects of the AHA! experience, it concluded with me learning more about the affective aspects of the AHA! experience. The details of my transformation in thinking about the phenomenon are revealed within the chapter alongside the data that precipitated this change in focus.

Chapter six details a study in which the survey that Jacques Hadamard used in his empirical study, some 50 years ago, is resurrected for the purpose of soliciting anecdotal accounts from contemporary mathematicians. This usage of Hadamard's survey produced a rich data set of accounts from some of the world's most prominent and well-respected mathematicians. These data are analysed for two purposes. The first of these is to distil from the mathematicans' accounts the very essence of the AHA! experience. The second is to formulate some conjectures as to what sorts of environmental and situational conditions are necessary for the occurrences of AHA! experiences. These conjectures are then used to structure the third study, which is presented in chapter seven.

The third study was designed to answer the question as to whether conditions surrounding a problem solving environment could be structured to increase the occurrences of AHA! experiences among students. This study necessitated the creation of a new form of student journaling for the purposes of recording and tracking these experiences. Both this form of journaling and the particulars of the environmental structuring are detailed in the chapter seven.

Although each of the studies presented in chapters five through chapter seven have their own related conclusions, these conclusions also speak to each other and serve to influence the characteristics of subsequent studies, as well as the analysis of the studies as a whole. The three studies, their independent conclusions, and the whole of the knowledge gained serve to produce some encompassing conclusions regarding the three research questions. These conclusions as well as contributions to the field of mathematics education research in general are presented in chapter eight. A theoretical contribution, a methodological contribution, and a pedagogical contribution of my research are outlined.

Finally, in chapter nine I explore the implications of these conclusions and contributions for the teaching and learning of mathematics in general. In particular I discuss the use and assessment of problem solving for the purposes of teaching mathematics content, as well as for teaching what it means to 'do' mathematics.

CHAPTER ONE
INVENTION, CREATIVITY AND THE AHA! EXPERIENCE

The AHA! experience is a term that is used to capture the essence of the experience of illumination in the context of 'doing' mathematics (Barnes, 2000; Burton, 1999a, 1999b; Davis & Hersh, 1980; Gardner, 1982, 1978; Mason, Burton, & Stacey, 1982; Perkins 2000). However, it is often the essence of the experience that exists within the literature and not the term itself (for example, Hadamard, 1945; Poincaré, 1952; Pólya, 1965/1981). It is through literature containing either the essence or the term that I come to describe the AHA!. Simply put, the AHA! experience is the *EXPERIENCE* of having an idea come to mind with "characteristics of brevity, suddenness, and immediate certainty" (Poincaré, 1952, p.54). It is the phenomenon of "sudden clarification" (Pólya, 1965/1981, p. 54) arriving in a "flash of insight" (Davis & Hersh, 1980, p. 283) and accompanied by feelings of certainty (Burton, 1999a; Fischbein, 1987) and bliss (Rota, 1997). In this chapter, as well as the next, I use the literature from a variety of different areas to construct a more comprehensive understanding of the AHA! experience. I begin by examining what it means to 'do' mathematics.

'Doing' Mathematics

Picture someone in the grips of 'doing' mathematics. There he sits, slightly unkempt, madly scribbling away on a pad of paper in some unintelligible language of symbols and diagrams (Kasner and Newman, 1940). The desk in his small office is in danger of being lost under the mountain of papers and books covered in similar scribbles. Occasionally he will break form his frantic writing to pace the room, mumbling to himself, perhaps making a few notations on a chalkboard before returning to his overburdened desk. This may be a Hollywood exaggeration of someone 'doing' mathematics but the image is not unfounded (see for example the case of Andrew Wiles and Fermat's Last Theorem in Singh, 1997).

My purpose for painting the above picture is not to comment on the eccentric nature of 'doing' mathematicians, but rather, to bring into question what exactly it is that the aforementioned person is 'doing'. Given the description, it is either a mathematician inventing new mathematics or it is a student solving a mathematical problem. The question then becomes, are they different? At the level of mathematics they are. A mathematician forging ahead into the uncharted territory of the mathematical landscape (Burton, 1999a; Sfard, 1994) is going where no one has gone before. A student diligently working at completing some challenging task set by his mathematics teacher is likely venturing down a much-travelled path. At the level of the individual, however, there is no difference. Both the research mathematician and the student are working in territory unfamiliar to them; "between the work of a student who tries to solve a problem in geometry or algebra and a work of invention, one can say there is only a difference of degree" (Hadamard, 1945, p.104)[4]. It is at this individual level, where mathematical invention (and creativity) and mathematical problem solving are indistinguishable from each other, that the 'doing' of mathematics happens (Silver, 1997), and it is at this level where the AHA! experience resides. I examine the literature on both of these aspects of 'doing' mathematics separately. The remainder of this chapter is dedicated to the review of the literature pertaining to the first of these two aspects, invention. In addition, I will examine the closely related phenomena of creativity, intuition, and imagination. Problem solving, the second aspect of 'doing' mathematics is dealt with in chapter two.

Mathematical Invention and Discovery

There are two theories regarding the origins of mathematics (Hersh, 1997). The first theory, attributed to Plato, states that mathematics exists independent of man's involvement in the field. As such, mathematical knowledge is 'discovered', much in the same way knowledge about the natural world is discovered. The second theory, referred to as the 'formalist theory', poses that mathematics is a construct of mankind, a product of human thinking. In this case, new knowledge in mathematics is 'invented', much like new technology is invented. Although, some mathematicians cling to one or the other of these theories as being the truth concerning the nature of mathematics and its origins, many find a middle

[4] This is in direct contrast to Resnick and Glaser (1976) who see problem solving and inventing as being distinct. The basis for this position is that for Resnick and Glaser invention involves creating something that did not exist before, whereas problem solving involves re-creating something that existed before.

ground that combines attributes of both. To see how these two extremes, along with the middle ground play out in practical terms consider the example of the infinitude of prime numbers. A Platonist would argue that the prime numbers existed independent and prior to us, and that they existed in infinitude. We merely discovered them. A Formalist would argue that prime numbers exist because we defined them as such. One possible middle ground between these two views is that the numbers themselves were invented (or defined), but the property that some numbers are prime was discovered, as was the fact that there are an infinite number of prime numbers.

This example highlights how it might be possible to draw a distinction between the usage of *invented* and *discovered*. In an attempt to draw further distinction it can be suggested that mathematics is *invented* when someone deliberately and wilfully creates something new, even if it is only new to that one individual. On the other hand, mathematics is *discovered* if by some mechanism a mathematical property emerges out of some already existing mathematical object (numeric, algebraic, geometric, or otherwise). So, for example, Napier invented logarithms, while Fermat discovered the theorem that bears his name. However, when definitions (which are invented) and properties (which are discovered) become conflated this distinction becomes convoluted. Hadamard (1945) offers an anecdote that nicely explains this conflation.

> *Such distinction has proved less evident than appears at first glance. Toricelli has observed that when one inverts a closed tube on the mercury trough, the mercury ascends to a certain determined height: this is a discovery; but, in doing this, he has invented the barometer; and there are plenty of examples of*

> *scientific results which are just as much discoveries as inventions. Franklin's invention of the lightning rod is hardly different from his discovery of the electric nature of thunder.* (p. xvii)

Hadamard goes on to state that because of this inability to make a clear distinction between discovery and invention he will not concern himself with it, and will, instead, treat them equally. Furthermore, he states that the "psychological conditions are quite the same for both cases" (p. xvii). For these very same reasons, the phenomena of discovery and invention will be treated within this chapter, as well as for the entire dissertation, without distinction.

There exists a large body of literature on mathematical invention, all of which stems from the work of two prominent French mathematicians from the first half of the 20th century. Henri Poincaré (1854–1912) and Jacques Hadamard (1865-1963) were well-established mathematicians who shared between them both a friendship as well as a curiosity regarding the origin of ideas and the creation of mathematics.

> *The genesis of mathematical creation is a problem which should intensely interest the psychologist. It is the activity in which the human mind seems to take the least from the outside world, in which it acts or seems to act only of itself and on itself, so that in studying the procedure of geometric thought we may hope to reach what is most essential in man's mind.* (Poincaré, 1952, p. 46)

They first posed, and subsequently formalized ideas regarding mathematical invention and creativity.

In 1908 Poincaré gave a presentation to the French Psychological Society in Paris entitled 'Mathematical Creation'. This presentation, as well as the essay it spawned, stands to this day as one of the most insightful, and thorough treatments of the topic of mathematical invention. In particular, the anecdote of Poincaré's own discovery of Fuschian function transformations stands as the most famous contemporary account of mathematical creation.

> *Just at this time, I left Caen, where I was living, to go on a geological excursion under the auspices of the School of Mines. The incident of the travel made me forget my mathematical work. Having reached Coutances, we entered an omnibus to go some place or other. At the moment when I put my foot on the step, the idea came to me, without anything in my former thoughts seeming to have paved the way for it, that the transformations I had used to define the Fuschian functions were identical with those of non-Euclidean geometry. I did not verify the idea; I should not have had the time, as, upon taking my seat in the omnibus, I went on with the conversation already commenced, but I felt a perfect certainty. On my return to Caen, for conscience' sake, I verified the results at my leisure.* (Poincaré, 1952, p. 53)

So powerful was his presentation, and so deep were his insights into his acts of invention that it could be said that he not so much described the characteristics of mathematical creativity, as defined them. From that point forth mathematical creativity, or even creativity in general, has not been discussed seriously without mention of Poincaré's name.

Inspired by this presentation Hadamard began his own empirical investigation into mathematical invention, the details of which are provided in chapter six. The results of this seminal work culminated in a series of lectures on mathematical invention at the École Libre des Hautes Etudes in New York City in 1943. These talks were subsequently published as The Psychology of Mathematical Invention in the Mathematical Field (Hadamard, 1945). Hadamard took the ideas that Poincaré had posed and, borrowing a conceptual framework for the characterization of the creative process, turned them into a stage theory.

This theory still stands as the most viable and reasonable description of the process of mathematical invention. Since then the work of many other mathematicians and educational researchers alike have only served to confirm Hadamard's (and Poincaré's) theory. In what follows I present this theory, referenced not only to Hadamard and Poincaré, but also to the many researchers whose work has informed and verified the different parts of the theory.

The phenomenon of mathematical invention, although marked by sudden illumination, consists of four separate stages stretched out over time, of which illumination is but one part. These stages are initiation, incubation, illumination, and verification (Hadamard, 1945). The first of these stages, the *initiation* phase, consists of deliberate and conscious work. This would constitute a person's voluntary, and seemingly fruitless, engagement with a problem and be characterized by an attempt to solve the problem by trolling through a repertoire of past experiences[5] (Bruner, 1964; Rusbult, 2000; Schön, 1987). This is an important part of the inventive process because it creates the tension of unresolved effort that sets up the conditions necessary for the ensuing emotional release at the moment of illumination (Barnes, 2000; Davis & Hersh, 1980; Feynman, 1999; Hadamard, 1945; Poincaré, 1952; Rota, 1997).

Following the initiation stage the solver, unable to come up with a solution stops working on the problem at a conscious level (Dewey, 1933) and begins to work on it at an unconscious level (Hadamard, 1945; Poincaré, 1952). This is referred to as the *incubation* stage of the inventive process and can last

[5] This is referred to as a process of 'design' (Rusbult, 2000), a characteristic of problem solving that will be discussed in greater detail in chapter two.

anywhere from several minutes to several years. After the period of incubation a rapid coming to mind of a solution, referred to as *illumination*, may occur. This is accompanied by a feeling of certainty (Poincaré, 1952) and positive emotions (Barnes, 2000; Burton 1999a, 1999b; Rota, 1997). Although the processes of incubation and illumination are shrouded behind the veil of the unconscious there are a number of things that can be deduced about them. First and foremost is the fact that unconscious work does, indeed, occur. Poincaré (1952), as well as Hadamard (1945), use the very real experience of illumination, a phenomenon that cannot be denied, as evidence of unconscious work, the fruits of which appear in the flash of illumination. No other theory seems viable in explaining the sudden appearance of solution during a walk, a shower, a conversation, upon waking, or at the instance of turning the conscious mind back to the problem after a period of rest (Poincaré, 1952). Also deducible is that unconscious work is inextricably linked to the conscious and intentional effort that precedes it.

> *There is another remark to be made about the conditions of this unconscious work: it is possible, and of a certainty it is only fruitful, if it is on the one hand preceded and on the other hand followed by a period of conscious work. These sudden inspirations never happen except after some days of voluntary effort which has appeared absolutely fruitless and whence nothing good seems to have come …* (Poincaré, 1952, p. 56)

Hence, the fruitless efforts of the initiation phase are only seemingly so. They not only set up the aforementioned tension responsible for the emotional release at the time of illumination, but also create the conditions necessary for the process to enter into the incubation phase.

With regards to the phenomenon of illumination, it is clear that this phase is the manifestation of a bridging that occurs between the unconscious mind and the conscious mind (Poincaré, 1952), a coming to (conscious) mind of an idea or solution. However, what brings the idea forward to consciousness is unclear. There are theories on aesthetic qualities of the idea (Poincaré, 1952; Sinclair, 2002), effective surprise/shock of recognition (Bruner, 1964), fluency of processing (Whittlesea & Williams, 2001), allowing the brain to rest (Helmholtz, cited in Krutetskii, 1976), shifting of attention (Mason, 1989), chance (Hadamard, 1945), or breaking functional fixedness (Ashcraft, 1989). For reasons of brevity I will only expand on the first of these.

Poincaré proposed that ideas that were stimulated during initiation remained stimulated during incubation. However, freed from the constraints of conscious thought and deliberate calculation, these ideas would begin to come together in rapid and random unions so that "their mutual impacts may produce new combinations" (Poincaré, 1952, p. 61). These new combinations, or ideas, would then be evaluated for viability using an aesthetic sieve (Sinclair, 2002), which allowed through to the conscious mind only the "right combinations" (Poincaré, 1952, p. 62). It is important to note, however, that good or aesthetic does not necessarily mean correct. Correctness is evaluated during the *verification* stage.

The purpose of verification is not only to check for correctness. It is also a method by which the solver re-engages with the problem at the level of details. That is, during the unconscious work the problem is engaged with at the level of ideas and concepts. During verification the solver can examine these ideas in closer details. Poincaré succinctly describes both of these purposes.

> *As for the calculations, themselves, they must be made in the second period of conscious work, that which follows the inspiration, that in which one verifies the results of this inspiration and deduces their consequences.* (Poincaré, 1952, p. 62)

Aside from presenting this aforementioned theory on invention Hadamard also engaged in a far-reaching discussion on a number of interesting, and sometimes quirky, aspects of invention and discovery that he had culled from the results of his empirical study, as well as from pertinent literature. This discussion was nicely summarized by James Newman (2000) in his commentary on the elusiveness of invention.

> *The celebrated phrenologist Gall said mathematical ability showed itself in a bump on the head, the location of which he specified. The psychologist Souriau, we are told, maintained that invention occurs by "pure chance", a valuable theory. It is often suggested that creative ideas are conjured up in "mathematical dreams", but this attractive hypothesis has not been verified. Hadamard reports that mathematicians were asked whether "noises" or "meteorological circumstances" helped or hindered research [..] Claude Bernard, the great physiologist, said that in order to invent "one must think aside". Hadamard says this is a profound insight; he also considers whether scientific invention may perhaps be improved by standing or sitting or by taking two baths in a row. Helmholtz and Poincaré worked sitting at a table; Hadamard's practice is to pace the room ("Legs are the wheels of thought", said Emile Angier); the chemist J. Teeple was the two-bath man.* (p. 2039)

Invention and discovery are not "part of the theories of logical forms" (Dewey, 1938, p.103). They are extra-logical. That is, discovery and invention are not representative of the lock-step logic and deductive reasoning that mathematics is often presumed to embody (Bibby, 2002; Burton, 1999b; Felix

Klein cited in Glas, 2002). Invention and discovery are part of a cohort of extra-logical processes that includes creativity, intuition, and imagination. In what follows I present literature on each of each of these processes in turn.

Creativity

The four stages of the AHA described above and characterized in Hadamard's seminal work were not original to Hadamard. A psychologist by the name of Wallace (Ashcraft, 1989) used the same characterization to describe the creative process about 20 years prior to Hadamard's work. Because of this close relationship between the two phenomena (invention and creativity) the body of literature on creativity is one that cannot be ignored. Aside from offering a variety of examples of AHA! experiences this literature also contributes to the construction of a better understanding of the AHA! experience through three distinct academic discourses[6]: the focus on product, process, and person. I provide a brief synopsis of each of these discourses. These synopses are then used in the last section of this chapter to arrive at a more general description of the AHA! experience.

'Creativity' is a term that can be used both loosely and precisely. That is, while there does exist a common usage of the term there also exists a tradition of academic discourse on the subject. A common usage of 'creative' refers to a process or a person whose products are original, novel, unusual, or even

[6] This is not to say that there exist only three discourses, but rather that I am focusing on only three discourses. Gardner (1993) does a good job of outlining a number of other approaches to creativity – such as psychometric, cognitive, personality/motivation, and historiometric - that are not going to be dealt with here.

abnormal (Bailin, 1994; Csikszentmihalyi, 1996). In such a usage, creativity is assessed on the basis of the external and observable products of the process, the process by which the product comes to be, or on the character traits of the person doing the 'creating'. Each of these usages are the roots of the discourses that I present here, the first of which concerns products.

Consider a mother who states that her daughter is creative because she drew an original picture. The basis of such a statement can lie either in the fact that the picture is unlike any the mother has ever seen, or unlike any her daughter has ever drawn before. This mother is assessing creativity on the basis of what her daughter has produced. However, the standards that form the basis of her assessment are neither consistent nor stringent. There does not exist a universal agreement as to what she is comparing the picture to (pictures by other children or other pictures by the same child). Likewise, there is no standard by which the actual quality of the picture is measured. The academic discourse that concerns assessment of products, on the other hand, is both consistent and stringent (Bailin, 1994; Csikszentmihalyi, 1996). This discourse concerns itself more with a fifth, and as yet unmentioned, stage of the creative process; *elaboration*. Elaboration is where inspiration becomes perspiration (Csikszentmihalyi, 1996). It is the act of turning a good idea into a finished product, and the finished product is ultimately what determines the 'creativity' of the process that spawned it (Getzels & Jackson, cited in Silver, 1997; Torrance, 1966); it cannot be a creative process if nothing is created (Bailin, 1994). In particular, this discourse demands that the product be assessed against other products within its field, by the members of that field, to determine if it is original AND useful. If it is, then the product is deemed to be creative. Note that such a use of assessment of end product pays very little attention to the actual process that brings this product forth.

The second discourse to be discussed concerns the creative process. The literature pertaining to this can be separated into two categories, a prescriptive discussion of the creativity process and a descriptive discussion of the creativity process. Although both of these discussions have their roots in the four stages that Wallace proposed makes up the creative process (Ashcraft, 1989), they make use of these stages in very different ways. The prescriptive discussion of the creative process is primarily focused on the first of the four stages, *initiation*, and is best summarized as a *cause-and-effect* discussion of creativity, where the thinking processes during the initiation stage are the *cause* and the creative outcome are the *effects* (Ghiselin, 1952). Some of the literature claims that the seeds of creativity lie in being able to think about a problem or situation analogically (Johnson-Laird, 1989). Other literature claims that utilizing specific thinking tools such as imagination, empathy, and embodiment (Root-Bernstein & Root-Bernstein, 1999) will lead to creative products. In all of these cases, the underlying theory is that the eventual presentation of a creative idea will be precipitated by the conscious and deliberate efforts during the initiation stage. On the other hand, the literature pertaining to a descriptive discussion of the creative process is inclusive of all four stages (Kneller, 1965; Koestler, 1964). For example, Csikszentmihalyi (1996), in his work on 'flow' attends to each of the stages, with much attention paid to the fluid area between conscious and unconscious work, or initiation and incubation. His claim is that the creative process is intimately connected to the enjoyment that exists during times of sincere and consuming engagement with a situation, the conditions of which he describes in great detail.

The third, and final, discourse on creativity that will be discussed here pertains to the person. This discourse is dominated by two distinct characteristics, habit (Bailin, 1994) and genius (Silver, 1997). Habit has to do with the personal habits as well as the habits of mind of people that have been deemed to be creative (Pehkonen, 1997). However, creative people are most easily identified through their reputation for genius (Silver, 1997). Consequently, this discourse is often dominated by the analyses of the habits of geniuses as is seen in the work of Ghiselin (1952), Koestler (1964), and Kneller (1965) who draw on historical personalities such as Albert Einstein, Henri Poincaré, Vincent Van Gogh, D. H. Lawrence, Samuel Taylor Coleridge, Igor Stravinsky, and Wolfgang Amadeus Mozart to name a few. The result of this sort of treatment is that creative acts are viewed as rare mental feats, which are produced by extraordinary individuals who use extraordinary thought processes (Weisburg, 1988).

Intuition

Intuition is an extra-logical process that, like invention and creativity, is an undeniable part of mathematics (Burton, 1999b; Hersh, 1997). In fact, intuition and creativity are strongly linked; "without intuition, there is no creativity in mathematics" (Wilder, cited in Burton, 1999b, p. 27). For these reasons, I present a synopsis of the literature pertaining to intuition.

At a very rudimentary level intuition can be thought of as a 'hunch' (Bruner, 1964). However, closer examination shows that there are, in fact, different types of intuitions, each of which is dependent on the role that it plays and the context within which it is invoked (Beth & Piaget, 1966; Fischbein, 1987). For example, in the process of problem solving, intuition may provide a direction to look in. In the case of examining a finished product, on the other hand, intuition can help to assess the relative worth of something. Fischbein (1987) claims that this is due to an expert's ability to consider non-salient features (including aesthetic elements) of the situation at an unconscious level. Consider the case when Hardy received the writings of Ramanujan. It was intuition, not careful conscious analysis of the work that told him that there was something meaningful in those papers (Hersh, 1997). Fischbein (1987) recognizes the functional dependency of intuition and classifies the different types according to their roles. He refers to a case of a hint as to where to look as *affirmatory intuition* and the rapid evaluation of something as *conjectural intuition.*

Imagination

Another extra-logical process is the imagination. It too has something to offer to the construction of an understanding of the AHA! experience. Imagination can be described as reaching out (Greene, 2000) along lines of conceivable trajectories as determined by one's own experiences (Dewey, 1933; Whitehead, 1959). To give substance to this statement an example is warranted. Imagine, for an instance, an animal that lives on a distant planet. What your imagination concocts may be unique and unseen, but not inconceivable. Most likely the creature of your imagination is rooted in some experience you've had (a real animal or a movie creature) along with some standard modifications (fangs, horns, stripes, extra limbs, etc.). Your imagination has reached out along conceivable trajectories. What is of relevance, however, is not the reaching out, but rather the mechanism by which plausibility of the creature is evaluated. Feasibility is evaluated at the conscious level while in the imagination it is evaluated at the unconscious level (Bruner, 1964) – relying on the process of conjectural intuition. As such, "imagination has the pragmatic value that it leaps ahead of the slow-moving caravan of well ordered thoughts and often scouts out reality long before its ponderous master" (Kasner & Newman, 1940). That is, the imagination relies heavily on the contributions of the unconscious mind as it creates and discards new ideas. The imagination is also inextricably linked to incubation and illumination which can be referred to as a leap of the imagination (Greene, 2000).

How Does the AHA! Experience Fit (or not fit)?

As mentioned earlier, the AHA! experience is the *EXPERIENCE* of having an idea come to mind with "characteristics of brevity, suddenness, and immediate certainty" (Poincaré, 1952, p. 54). This description is meant to capture the essence of the experience. In this section I draw from the literature presented above to move from this description of the AHA! experience to a more tightly constructed understanding of what is meant by the AHA! that shows how it is similar to, yet different from, invention, discovery, creativity, intuition, and imagination. Before I do, however, I wish to point out that my goal is not to create a definition by which I can include and exclude experiences as AHA!'s, but rather to present a description with which to discuss such experiences. As a consequence, the description that will emerge is not meant to be precise or definitive, only adequate.

First and foremost, the AHA! experience is illumination, the rapid coming to mind of an idea. However, it is also much more than this, for ideas come to mind all the time, quickly, and without clear understanding from whence it comes. Hadamard (1945) uses speech as an example of this phenomenon. He states that as he speaks, the next word or even the next sentence is always at the ready, seemingly without conscious effort. This is not to say that speech is an instance of illumination, but rather that illumination, like the AHA! experience, is heavily dependent on what has come before. That is, initiation and incubation are requisite for illumination, and so too are they requisite for the AHA!. Verification only serves to confirm and validate the idea that comes forth during the illumination. It does not, however, change whether or not illumination happens. Neither does it change the intensity of the illumination. Upon retrospection, however, the eventual significance of the insight may serve to strengthen or weaken the memory of the event (Ashcraft, 1976). It may even change the memory of the event (Whittlesea & Wright, 1997), but it cannot change the actual experience of the event. This takes into consideration the phenomenon that can be referred to as a *false AHA!*. A false AHA! is an AHA! experience in which the insight that comes during illumination is incorrect. This does not eliminate the fact that the solver has experienced an AHA!, even though that AHA! was fruitless. The AHA! experience concerns itself only with the first three stages of the inventive process: initiation, incubation, and illumination.

Likewise, the AHA! experience is concerned only with these same three stages of the creative process. That is, the AHA! experience is the creative process without any concern for the originality, usefulness, or even existence of a product. Depending on which discourse on creativity that one subscribes to this may or may not mean that the AHA! experience is synonymous with a creative experience. This is not an obvious distinction primarily because almost all of the literature on creativity uses anecdotal accounts of AHA! experiences as evidence of creativity. Regardless, the AHA! experience is part of the creative process, a process that may be influenced by personal habits and habits of mind, but should not be restricted only to those with a 'genius' status.

With regards to intuition and imagination, the AHA! experience shares with these phenomena a kinship. Intuition may very well be the one mechanism that remains constant in both the conscious and the unconscious workings on a problem. It has the ability to asses the feasibility of an idea through a process that lies outside of the "theories of logical forms" (Dewey, 1938, p.103), attending to salient as well as non-salient features alike (Fischbein, 1987). Likewise, the imagination incorporates aspects of the extra-logical in its ability to evaluate the feasibility of new ideas quickly (Bruner, 1964; Hersh, 1997; Kasner & Newman, 1940). Furthermore, the imagination may be a necessary, yet not sufficient precursor to invention.

> *Invention presupposes and should not be confused with it. For the act of invention implies the necessity of a lucky find and of achieving full realization of this find. What we imagine does not necessarily take on a concrete form and may remain in a state of virtuality, whereas invention is not conceivable apart from its actually being worked out. Thus, what concerns us here is not imagination in itself, but rather creative imagination: the faculty that helps us pass from the level of conception to the level of realization.* (Igor Stravinsky cited in Root-Bernstein & Root-Bernstein, 1999, p. x)

In summary, the AHA! experience encompasses all that leads up to illumination in the process of invention, discovery, and creativity with no consideration for the validity of the ensuing insight. It begins with the initiation phase during which the solver attacks the problem intentionally and directly, relying on past experiences, intuition, and imagination in the selection and evaluation of directions of attack. This wilful effort then wanes as the process gives itself over to the incubation phase during which time the conscious mind of the solver is distracted away from the problem. This is followed by illumination where an idea as to the solution or method towards a solution suddenly appears, filling the solver with a sense of certainty, relief, and joy.

CHAPTER TWO
PROBLEM SOLVING

> *My attitude to mathematics has changed radically since I was a student. At that time I though of mathematics as a body of theorems, a static concept. I learned later to look at it as a problem solving activity. The theorems are still important, but perhaps less so nowadays.*
>
> \- Ulf Grenander[7]

Mathematical problem solving is synonymous with invention and discovery and, as such, it has a close relationship with the AHA! experience. In this chapter I will use the literature on problem solving to examine and discuss this relationship. I begin by presenting the most general of all problem solving heuristics referred to as *problem solving by design*; a heuristic that is entirely based on the resources of past experience and prior knowledge. I then present two specific problem solving heuristics that refine the principles of problem solving by design, and three heuristics that extend these principles to acknowledge the role of the AHA! experience in the problem solving process.

[7] Ulf Grenander participated in the study that I present in chapter six. This is a quote from that study.

Problem Solving by Design

In a general sense, *design* is defined as the algorithmic and deductive approach to solving a problem (Rusbult, 2000). This is the process that is used during the initiation phase of invention and creativity discussed in the previous chapter. The process begins with a clearly defined goal or objective after which there is a great reliance on relevant past experience, referred to as repertoire (Bruner, 1964; Schön, 1987), to produce possible options that will lead towards a solution of the problem (Poincaré, 1952). These options are then examined through a process of conscious evaluations (Dewey, 1933) to determine their suitability for advancing the problem towards the final goal. In very simple terms, problem solving by design is the process of deducing the solution from that which is already known.

Mayer (1982), Schoenfeld (1982), and Silver (1982) state that prior knowledge is a key element in the problem solving process. Prior knowledge influences the problem solver's understanding of the problem as well as the choice of strategies that will be called upon in trying to solve the problem. In fact, prior knowledge and prior experiences is ALL that a solver has to draw on when first attacking a problem. As a result, all problem solving heuristics must incorporate this resource of past experiences and prior knowledge into their initial attack on a problem, and all do. Some heuristics refine these ideas, and some heuristics extend them. Of the heuristics that refine, none is more influential than the one created by George Pólya (1887 – 1985).

George Pólya: How to Solve It

In his book How to Solve It (1957) Pólya lays out a problem solving heuristic that relies heavily on a repertoire of past experience. He summarizes the four-step process of his heuristic as follows:

1. UNDERSTANDING THE PROBLEM

- ***First.*** *You have to understand the problem.*
- *What is the unknown? What are the data? What is the condition?*
- *Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?*
- *Draw a figure. Introduce suitable notation.*
- *Separate the various parts of the condition. Can you write them down?*

2. DEVISING A PLAN

- ***Second.*** *Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a plan of the solution.*
- *Have you seen it before? Or have you seen the same problem in a slightly different form?*
- *Do you know a related problem? Do you know a theorem that could be useful?*
- *Look at the unknown! And try to think of a familiar problem having the same or a similar unknown.*
- *Here is a problem related to yours and solved before. Could you use it? Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?*
- *Could you restate the problem? Could you restate it still differently? Go back to definitions.*
- *If you cannot solve the proposed problem try to solve first some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or data, or both if*

necessary, so that the new unknown and the new data are nearer to each other?

- *Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?*

3. CARRYING OUT THE PLAN

- ***Third.** Carry out your plan.*
- *Carrying out your plan of the solution, check each step. Can you see clearly that the step is correct? Can you prove that it is correct?*

4. LOOKING BACK

- ***Fourth.** Examine the solution obtained.*
- *Can you check the result? Can you check the argument?*
- *Can you derive the solution differently? Can you see it at a glance?*
- *Can you use the result, or the method, for some other problem?*

The emphasis on *auxiliary problems, related problems,* and *analogous problems* that are, in themselves, also *familiar problems* is an explicit manifestation of relying on a repertoire of past experience to solve problems. This use of familiar problems also requires an ability to deduce from these related problems a recognizable and relevant attribute that will transfer to the problem at hand. The mechanism that allows for this transfer of knowledge between analogous problems is known as analogical reasoning (English, 1998, 1997; Novick, 1995, 1990, 1988; Novick & Holyoak, 1991) and has been shown to be an effective, but not always accessible, thinking strategy[8].

[8] What binds problems, and solution strategies, together may be concrete mathematical structures, but these are often masked by the much more obvious, and less useful, surface structures. For example, noticing that something is an optimization problem does not really help in solving it because such problems encompass a large collection of problems with different mathematical structures. Conversely, focusing on the deeper mathematical structures does not accomplish much either. For example, the large collections that can be solved using the 'pigeon hole' principle are so varied that they are rarely associated with one another.

Step four in Pólya's heuristic, *looking back*, is also a manifestation of utilizing prior knowledge to solve problems, albeit an implicit one. *Looking back* makes connections "in memory to previously acquired knowledge [..] and further establishes knowledge in long-term memory that may be elaborated in later problem-solving encounters" (Silver, 1982, p. 20). That is, *looking back* is a forward-looking investment into future problem solving encounters, it sets up connections that may later be needed.

Pólya's heuristic is a refinement on the principles of *problem solving by design*. It not only makes explicit the focus on past experiences and prior knowledge, but also presents these ideas in a very succinct, digestible, and teachable manner. This heuristic has become a popular, if not the most popular, mechanism by which problem solving is taught and learned.

Alan Schoenfeld: Mathematical Problem Solving

The work of Shoenfeld is also a refinement on the principles of *problem solving by design*. However, unlike Pólya who refined these principles at a theoretical level, Schoenfeld has refined them at a practical and empirical level. That is, he has not only thought about problem solving, but he has researched it. In addition to studying taught problem solving strategies he has also managed to identify and classify a variety of strategies, mostly ineffectual, that students invoke naturally (Schoenfeld, 1994, 1985). In so doing, he has created a better understanding of how students solve problems, as well as a better understanding of how problems should be solved and how problem solving should be taught.

For Schoenfeld, the problem solving process is ultimately a dialogue between the problem solver's prior knowledge, his attempts, and his thoughts along the way (Schoenfeld, 1982). As such, the solution path of a problem is an emerging and contextually dependent process. This is a departure from the predefined and contextually independent processes of Pólya's heuristic. This can be seen in Schoenfeld's description of a good problem solver.

> *To examine what accounts for expertise in problem solving, you would have to give the expert a problem for which he does not have access to a solution schema. His behavior in such circumstances is radically different from what you would see when he works on routine or familiar "non-routine" problems. On the surface his performance is no longer proficient; it may even seem clumsy. Without access to a solution schema, he has no clear indication of how to start. He may not fully understand the problem, and may simply "explore it for a while until he feels comfortable with it. He will probably try to "match" it to familiar problems, in the hope it can be transformed into a (nearly) schema-driven solution. He will bring up a variety of plausible things: related facts, related problems, tentative approaches, etc. All of these will have to be juggled and balanced. He may make an attempt solving it in a particular way, and then back off. He may try two or three things for a couple of minutes and then decide which to pursue. In the midst of pursuing one direction he may go back and say "that's harder than it should be" and try something else. Or, after the comment, he may continue in the same direction. With luck, after some aborted attempts, he will solve the problem.* (p. 32-33)

Aside from demonstrating the emergent nature of the problem solving process, this passage also brings forth two consequences of Schoenfeld's work. The first of these is the existence of problems for which the solver does not have "access to a solution schema". Unlike Pólya, who's heuristic is a 'one size fits all (problems)' heuristic, Schoenfeld acknowledges that problem solving heuristics are, in fact, personal entities that are dependent on the solver's prior knowledge as well as their understanding of the problem at hand. Hence, the problems that a person can solve through his or her personal heuristic are finite and limited.

The second consequence that emerges from the above passage is that if a person lacks the solution schema to solve a given problem s/he may still solve the problem with the help of *luck*. This is an acknowledgement, if only indirectly so, of the difference between problem solving in an intentional and mechanical fashion verses problem solving in a more creative fashion, which is neither intentional nor mechanical (Pehkonen, 1997). Although the heuristics that have been dealt with thus far are all intentional and mechanical in nature that is not to say that all heuristics rely only on the logical processes of problem solving. In the next three sections heuristics that take into account the contribution of the extra-logical processes of the AHA! experience will be discussed

David Perkins: Breakthrough Thinking

Many consider a problem that can be solved by intentional and mechanical means to not be worthy of the title 'problem'. Resnick and Glaser (1976) define a problem as being something that you do NOT have the experience to solve. As such, a repertoire of past experiences sufficient for dealing with such a 'problem' would disqualify it from the ranks of 'problems' and relegate it to that of 'exercises'. For a problem to be classified as a 'problem' it must be 'problematic'. Although such an argument is circular it is also effective in expressing the ontology of mathematical 'problems'.

Any problem in which you can see how to attack it by deliberate effort, is a routine problem, and cannot be an important discovery. You must try and fail by deliberate efforts, and then rely on a sudden inspiration or intuition, or if you prefer to call it luck.

- Dan J. Kleitman[9]

David Perkins also requires problems to be problematic. The work presented in his book *Archimedes' Bathtub: The Art and Logic of Breakthrough Thinking* (2000) deals with situations in which the solver has gotten stuck and no amount of intentional or mechanical adherence to the principles of past experience and prior knowledge is going to get them unstuck. That is, he deals with problems that, by definition, cannot be solved through a process of *design* (or through the heuristics proposed by Pólya and Schoenfeld). Instead, the solver must rely on the extra-logical process of what Perkins calls *breakthrough thinking*, a process identical to that of an AHA! experience, to get them through.

Perkins begins by distinguishing between *reasonable* and *unreasonable* problems. Although both are solvable, only reasonable problems are solvable through reasoning. Unreasonable problems require a breakthrough in order to solve them. However, the problem itself is inert, it is neither reasonable nor unreasonable, that quality is brought to the problem by the solver. That is, if a student cannot solve a problem by direct effort then *that* problem is deemed to be unreasonable for *that* student. Perkins also acknowledges that what is an unreasonable problem for one person is a perfectly reasonable problem for another person. That is, reasonableness is dependent on the person.

[9] Dan J. Kleitman participated in the study that I present in chapter six. This is a quote from that study.

This is not to say that, once found, the solution cannot be seen as accessible through reason. During the actual process of solving, however, direct and deductive reasoning does not work. Perkins uses several classic examples to demonstrate this, the most famous being the problem of connecting nine dots in a 3 x 3 array with four straight lines without removing pencil from paper, the solution to which is presented in figure 3, below.

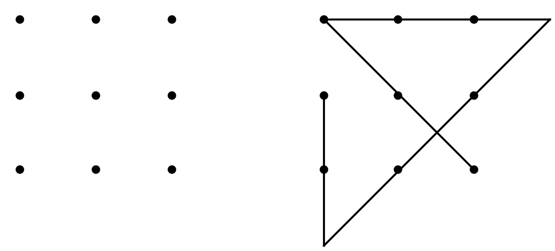

Figure 3: Nine Dots – Four Lines Problem and Solution

To solve this problem, Perkins claims that the solver must recognize that the constraint of staying within the square created by the 3 x 3 array is a self-imposed constraint. He further claims that until this is recognized no amount of reasoning is going to solve the problem. That is, at this point in the problem solving process the problem is unreasonable. However, once this self-imposed constraint is recognized the problem, and the solution, are perfectly reasonable. Thus, the solution of an, initially, unreasonable problem is reasonable.

The problem solving heuristic that Perkins has constructed to deal with solvable, but unreasonable, problems revolves around the idea of breakthrough thinking and what he calls *breakthrough problems*. A *breakthrough problem* is a solvable problem in which the solver has gotten stuck and will require an AHA! to get unstuck and solve the problem. Perkins poses that there are only four types of unreasonable problems, which he has named *wilderness of possibilities, the clueless plateau, narrow canyon of exploration,* and *oasis of false promise*. The names for the first three of these types of problems are related to the Klondike gold rush in Alaska, a time and place in which gold was found more by luck than by direct and systematic searching.

The *wilderness of possibilities* is a term Perkins has given to a problem that has many tempting directions but few actual solutions. This is akin to a prospector searching for gold in the Klondike. There is a great wilderness in which to search, but very little gold to be found. The *clueless plateau* is given to problems that present the solver with few, if any, clues as to how to solve it. The *narrow canyon of exploration* is used to describe a problem that has become constrained in such a way that no solution now exists. The nine-dot problem presented above is such a problem. The imposed constraint that the lines must lie within the square created by the array makes a solution impossible. This is identical to the metaphor of a prospector searching for gold within a canyon where no gold exists. The final type of problem gets its name from the desert. An *oasis of false promise* is a problem that allows the solver to quickly get a solution that is close to the desired outcome; thereby tempting them to remain fixed on the strategy that they used to get this almost-answer. The problem is, that like the canyon, the solution does not exist at the oasis; the solution strategy that produced an almost-answer is incapable of producing a complete answer.

Likewise, a desert oasis is a false promise in that it is only a reprieve from the desolation of the dessert and not a final destination.

Believing that there are only four ways to get stuck, Perkins has designed a problem solving heuristic that will up the chances of getting unstuck. This heuristic is based on what he refers to as "the logic of lucking out" (p. 44) and is built on the idea of introspection. By first recognizing that they are stuck, and then recognizing that the reason they are stuck can only be attributed to one of four reasons, the solver can access four strategies for getting unstuck, one each for the type of problem they are dealing with. If the reason they are stuck is because they are faced with a *wilderness of possibilities* they are to begin roaming far, wide, and systematically in the hope of reducing the possible solution space to one that is more manageable. If they find themselves on a *clueless plateau* they are to begin looking for clues, often in the wording of the problem. When stuck in a narrow canyon of possibilities they need to re-examine the problem and see if they have imposed any constraints. Finally, when in an *oasis of false promise* they need to re-attack the problem in such a way that they stay away from the oasis.

Of course, there are nuances and details associated with each of these types of problems and the strategies for dealing with them. However, nowhere within these details is there mention of the main difficulty inherent in introspection; that it is much easier for the solver to get stuck than it is for them to recognize that they are stuck. Once recognized, however, the details of Perkins' heuristic offer the solver some ways for recognizing why they are stuck.

John Mason: Thinking Mathematically

The work of John Mason, Leone Burton, and Kaye Stacey in their book *Thinking Mathematically* (1982) also recognizes the fact that for each individual there exists problems that will not yield to their intentional and mechanical attack. The heuristic that they present for dealing with this has two main processes with a number of smaller phases, rubrics, and states. The main processes are what they refer to as *specializing* and *generalizing*. Specializing is the process of getting to know the problem and how it behaves through the examination of special instances of the problem. This process is synonymous with problem solving by design and involves the repeated oscillation between the *entry* and *attack* phases of Mason et al.'s heuristic. The entry phase is comprised of 'getting started' and 'getting involved' with the problem by using what is immediately known about it. Attacking the problem involves conjecturing and testing a number of hypotheses in an attempt to gain greater understanding of the problem and to move towards a solution.

At some point within this process of oscillating between entry and attack the solver will get stuck, which Mason et al. refer to as "an honourable and positive state, from which much can be learned" (p. 55). The authors dedicate an entire chapter to this state in which they acknowledge that getting stuck occurs long before an awareness of being stuck develops. They proposes that the first step to dealing with being stuck is the simple act of writing STUCK!

> *The act of expressing my feelings helps to distance me from my state of being stuck. It frees me from incapacitating emotions and reminds me of actions that I can take.* (p. 56)

The next step is to reengage the problem by examining the details of what is known, what is wanted, what can be introduced into the problem, and what has been introduced into the problem (imposed assumptions). This process is engaged in until an AHA! which advances the problem towards a solution is encountered. If, at this point, the problem is not completely solved the oscillation is then resumed.

At some point in this process an attack on the problem will yield a solution and generalizing can begin. Generalizing is the process by which the specifics of a solution are examined and questions as to why it worked are investigated. This process is synonymous with the verification and elaboration stages of invention and creativity. Generalization may also include a phase of review that is similar to Pólya's *looking back*.

Gestalt: The Psychology of Problem Solving

The Gestalt psychology of learning believes that all learning is based on insights (Koestler, 1964). This psychology emerged as a response to behaviourism, which claimed that all learning was a response to external stimuli. Gestalt psychologists, on the other hand, believed that there was a cognitive process involved in learning as well. With regards to problem solving, the Gestalt school stands firm on the belief that problem solving, like learning, is a product of insight and as such, cannot be taught. In fact, the theory is that not only can problem solving not be taught, but also that attempting to adhere to any sort of heuristic will impede the working out of a correct solution (Krutetskii, 1976). Thus, there exists no Gestalt problem solving heuristic. Instead, the practice is to focus on the problem and the solution rather than on the process of coming up with a solution. Problems are solved by turning the problem over and over in the mind until an insight, a viable avenue of attack, presents itself. At the same time, however, there is a great reliance on prior knowledge and past experiences. The Gestalt method of problem solving is at the same time very different and very similar to the process of design.

Gestalt psychology has not fared well during the evolution of cognitive psychology. Although it honours the work of the unconscious mind it does so at the expense of practicality. If learning is, indeed, entirely based on insight then there is little point in continuing to study learning. "When one begins by assuming that the most important cognitive phenomena are inaccessible, there really is not much left to talk about" (Schoenfeld, 1985, p. 273). However, of interest here is the Gestalt psychologists' claim that focus on problem solving methods creates functional fixedness (Ashcraft, 1989). Mason et al. (1982), as well as Perkins (2000) deal with this in their work on getting unstuck.

Mathematical Problem Solving: Perceptions Versus Reality

Mathematics has often been characterized as the most precise of all sciences. It has earned this title primarily because of its use of deductive logic and formal proof and it now suffers from a misconception that this is all that mathematics is (Hanna, 1989). Lost in such a misconception is the fact that mathematics often has its roots in the fires of creativity, being born of the extra-logical processes of illumination and intuition. However, once created it is "encoded in a linear textual format born out of the logical formalist practice that now dominates mathematics" (Borwein & Jörgenson, 2001). Likewise, problem solving heuristics that are based solely on the processes of logical and deductive reasoning distort the true nature of problem solving. Certainly, there are problems in which logical deductive reasoning is sufficient for finding a solution. However, as presented in this chapter there are also solvable problems for which such logical and deductive reasoning is not sufficient. In such instances there needs to be accommodations made for the extra-logical processes of insight, illumination, and AHA! experiences in order to produce solutions.

Fortunately, as elusive as such processes are, there does exist problem solving heuristics that incorporate them into their strategies. Heuristics such as those by Perkins (2000) and Mason et al. (1982) have found a way of combining the intentional and mechanical processes of problem solving by design with the extra-logical processes of the AHA! experience. Furthermore, they have managed to do so without having to fully comprehend the inner workings of this mysterious process.

CHAPTER THREE
RESEARCH QUESTIONS

Having examined the AHA! experience, first in the general context of the extra-logical phenomena of invention, discovery, creativity, intuition, and imagination, and second in the specific context of mathematical problem solving it has become clear to me that there are many questions regarding the phenomenon that remain unanswered by the literature. Many of these have to do with the incubation and illumination process of mathematical discovery. These processes are cloaked in mystery in that they are, for the most part, unobservable. For example, what sort of thought processes are going on during the incubation period? Was Poincaré (1952) correct in his description of ideas crashing together in random occurrences? By what mechanism is a solution moved from the unconscious mind to the conscious mind? Do all unresolved problems incubate? Such problems are of great interest, and "should intensely interest the psychologists" (p. 46). However, I am not a psychologist. That is not to say that I am ambivalent regarding such questions, for I am not, but there are other questions that concern me more.

My questions have more to do with the nature of the AHA! experience itself, and how it compares to other mathematical experiences. I am interested not just in learning about the AHA! but in applying it, in using it as a pedagogical tool. Consequently, I have three research questions that are in line with this way of thinking, the answers to which do not exist in the literature.

Research Question One

What is the essence of the AHA! experience?

It is clear that in the context of doing mathematics there are experiences that can be referred to as AHA! experiences and experiences that cannot. What is not clear, however, is what it is about those experiences that can be referred to as AHA! experiences that sets them apart from other mathematical experiences. This is what I am interested in determining. However, I am not interested in answering this question at the level of words[10]. That is, I do not wish to find or create distinction through the treatment of definitions and the imposition of criteria. I want to find the answer within the phenomenon itself. I want to find the *essence* of the AHA! experience.

This is perhaps a little intangible for a research question, but it is very much representative of my interest and my thinking on the topic. Consider for a moment a student who has experienced an AHA! Do they have to be educated about it in order to know that they have had one? Do they have to know the definition to be able to discern this phenomenon from other experiences. I argue that they do not. They know they have had an AHA! experience because they have had it. The phenomenon is self-actualizing; it defines itself. I want to honour this quality of the phenomenon in my research. I want to pull from the

[10] "At the level of words, there are really no new ideas. Good results do not come from inventing new words." (Dan J. Kleitman, excerpt from research data, see chapter six).

phenomenon that which is common to all AHA! experiences. I want to examine the AHA! experience among several different populations, from students to mathematicians, and see how the phenomenon is the same and how it is different.

Research Question Two

What is the effect of an AHA! experience on a learner?

It is generally understood that with the successful resolution of a problematic situation some learning and deeper understanding is achieved. This is the premise behind discovery learning (Dewey, 1916; Bruner, 1961), constructivism (Steffe, Cobb, & von Glasersfeld, 1988) as well as teaching through problem solving (Cobb, Wood, & Yackel, 1991) and problem posing (Boaler, 1997; Brown & Walter, 1983). With the strong connection between the AHA! experience and the resolution of problems it is only natural to assume that these effects on learning and understanding would be realized in the context of an AHA! experience as well. The question is, is this really the case?

However, there is another dimension to the learner that needs to be considered in asking this question. This other dimension is that of the learner's feelings, how they feel about mathematics and how they feel about their abilities to do mathematics. Although the research question, as stated, does not exclude the consideration of this dimension it does not explicitly include it either. As such, I emphasise this second dimension here and ask the question: What is the effect of an AHA! experience on a learner's *feelings*?

One further dimension that this question can be explored along pertains to our understanding of *learner*. Up until now this has implied a student. However, students are not the only learners of mathematics. Mathematicians too, to a great extent, are acting as learners when they are struggling to solve a mathematical problem. Are their experiences with the AHA! any different than those of a student? In particular, what is the effect of an AHA! experience on a mathematician's cognitive and affective domains?

Research Question Three

Can the AHA! experience be controlled, and if so can it be invoked?

In many ways this was the reason I became interested in the AHA! experience. If the AHA! does have a positive effect on a learner's cognitive and/or affective domains then the ability to control the phenomenon would have powerful and wide ranging implications for teaching and learning. The previous two questions are in place in order to provide the background knowledge required to pursue these implications. Research question one is designed to understand the phenomenon better, research question two is designed to measure its effect, and research question three is designed to test its applicability. Can the descriptive understanding of the AHA! experience be translated into a prescriptive one? Can the AHA! experience be controlled?

Summary of Research Questions

In many ways the three research questions are representative of a progression from the theoretical to the practical. In the chapters that follow the answers to these questions will be pursued through a number of different empirical studies. However, the questions are not explored in isolation from each other. They are ever present, and the research continually serves to give greater insight into each of them. As the results of the individual studies are presented I will also present conclusions as to how that particular study informs my understanding regarding some or all of these three questions. In the end, these conclusions, along with an understanding that comes from being able to reflect back on all the studies simultaneously, combine to form my final conclusions regarding the three research questions.

Between this point and the conclusions, however, are four chapters representing the story of my coming to understand the AHA! experience, and it is very much a story. I present it in chronological order and maintain somewhat of a narrative style in its telling. The outline for this story is laid out in the next chapter, *The Journey: An Overview of Research Methodology*, and the details are presented in the subsequent three chapters, each presenting a different study.

CHAPTER FOUR
THE JOURNEY: AN OVERVIEW OF RESEARCH METHODOLOGY

The three research questions aside, a more pressing, and perhaps more difficult question is: How is one going to collect meaningful data on a phenomenon as rare and as fleeting as the AHA! experience? In the early stages of my research I naively believed that I could accomplish this in a clinical interview setting. I thought that through the orchestration of the mathematical stimuli that the participant received I would be able to induce AHA! experiences that could then be captured using audio and/or video tape. So, I attempted to use a clinical interview setting to orchestrate the occurrence of an AHA! experience in 12 participants (Liljedahl 2002, Liljedahl 2001). I presented each participant with two mathematical problems to be solved (see appendix C). Although appearing to be very different on the surface, these problems were analogous in that the underlying mathematical structures of the problems were isomorphic. My hope was that, after having solved the first problem, and after having given the second problem some thought, the participant would suddenly see the relationship between the two problems and have an AHA! experience. In the end, I was unsuccessful; none of the participants displayed any signs of having had an AHA! experience. However, in using a framework of analogical reasoning to interpret the data I began to see problems with my research design and the assumptions that it was built upon.

First and foremost, my plans to orchestrate such events were built on the erroneous assumption that the participants would interpret the mathematical stimuli that I was providing in the way that I was intending. I now know how wrong such an assumption is. Interpretation is an important dimension of the assimilation of mathematical understanding (English, 1998, 1997; Novick, 1995, 1990, 1988; Novick & Holyoak, 1991; Peirce, 1955), and the questions I was using left much to be interpreted. Had the participants seen the mathematics the way I had intended, then they would have seen the similarity in the questions. This was not the case. In fact, of the 12 participants none saw any similarity between the two questions, including the six participants who solved both problems using the exact same strategy. This brought me to another erroneous assumption.

I had originally believed that research participants' actions were reflective of their thinking. Of course this is true, but what is not true is that an outside observer will necessarily interpret these actions correctly. During this initial study six of the 12 participants were able to solve both problems correctly and completely. Furthermore, each of these six participants used an identical process for solving the second problem as they did in solving the first. I initially interpreted this to mean that they were using what they had learned from the first problem to solve the second problem; that they were using analogical reasoning (English, 1998, 1997; Novick, 1995, 1990, 1988; Novick & Holyoak, 1991). However, as already mentioned, none of the six participants saw any similarity between the two problems, even when prompted to look for it. I realized that I would never be able to rely solely on my interpretation of participants' actions as a source for any meaningful data.

A further flaw in my experimental design was the role that the environment and setting play in the facilitation of AHA! experiences. Upon reflection, I now see that the clinical interview is not at all conducive to the fostering of such phenomena. As I discuss in greater detail in chapters seven, eight, and nine, a person needs room to move, things to distract them, and people to interact with. The clinical interview provides none of these for the participants. However, were I to provide a more open and free environment for the participants to engage in problem solving I would drastically reduce my ability to track their mathematical interactions and stimuli.

Finally, I have come to see that, regardless of setting, the real-time capture of an AHA! experience (in its entirety) is a very difficult task. There are three main reasons for this. They are, in no particular order: TIME, TIME, and TIME. That is, time plays an incredibly important role in the entire process of mathematical problem solving. To begin with, the boundaries are fuzzy. When does a person first become exposed to a problem? If it is a student working on an assigned task this may be easily discernable, but if it is with regards to the spontaneous exploration of mathematics, perhaps out of curiosity or a desire to create new mathematics, it is not always clear. When is s/he incubating and when is s/he merely not consciously working on it? When is s/he finished? Such boundaries are often indiscernible, and usually unobservable. Even if one chooses to ignore such boundary issues, the simple task of observing someone from the moment a well-defined problem is given to the time it is solved would require a great deal of tracking of non-directed activity. As discussed in chapter one, people do not spend their time during incubation sitting in a chair NOT thinking. They do other things; think other things. The 20 minutes (or 40 minutes, or 60 minutes) allotted for problem solving does not allow for the distractions of life to help with the incubation process. The AHA! experience is much more than a single moment in time. It is part of a much larger set of experiences stretched over a long period of time, with many interruptions, deviations, and distractions.

All in all, this initial attempt to generate some empirical data with regards to AHA! experiences was an abysmal failure. There were no AHA!'s. There were no great insights into the inner workings of the participants' minds. However, there were insights into my research design. I learned a great deal about the limitations of the clinical interview for the purposes of capturing AHA! experiences.

A Look Towards Anecdotal Data

Given my initial difficulties, presented above, I concluded that I had to look elsewhere for my empirical data. I decided to explore the possibility of using anecdotal accounts that could provide me with descriptions of the AHA! experience itself. How detailed these descriptions were would determine how well I would be able to look into the heart of the experience, to see what lay at its core. Furthermore, reflective anecdotal accounts had the added benefit of creating an instrument by which the participant can articulate what happened, both from the perspective of the external stimuli and actions, as well as the more elusive internal interpretations, decisions, and discoveries. My participants would be able to tell me what they did and what they thought, something my initial efforts with clinical interviews had failed to do. However, there were details to work out, and hurdles to overcome before I could begin to collect and analyse such data.

The first detail in need of attention was deciding what, exactly, such data could look like. Anecdotal accounts of mathematical experiences can take on many different forms. My review of the literature on mathematical discovery and creativity had revealed that reflective anecdotal accounting was a good source of empirical data. Research mathematicians had been quite forthcoming, and seemingly sincere, with detailed accounts of their AHA! experiences both from interviews (Burton, 1999a, 1999b; Sfard, 1994) and surveys (Hadamard, 1945). Likewise, mathematics students had been equally forthcoming, if not quite as articulate, about their experiences (Barnes, 2000). So, I decided I would attempt this form of data collection in my research.

Reflective Anecdotal Accounting: Undergraduate Students

In my first attempt to collect anecdotal data I used a survey type format to collect data from a group of undergraduate mathematics students enrolled in a

'Foundations of Mathematics for Teachers' course. The participants were asked to recount an AHA! experience that they had had. Although I will leave the details of the survey, how it was administered, and the result for greater discussion in chapter five, I will say now that it was a great success.

Reflective Anecdotal Accounting: Prominent Mathematicians

Inspired by the success of this format I set my goals on a group of participants I felt would provide me with even greater insight into their AHA! experiences. I decided I was going to survey the top mathematicians in the world. To this end I enlisted the aid of Peter Borwein. This was a lofty goal, to be sure, and if it were to succeed we would have to be very careful in how we proceeded. The survey, which we had decided to send to mathematicians via email, had to be enticing enough to attract their attention, short enough to keep their attention, and thorough enough to provide me with a rich set of data. In the end I received 25 responses from some of the most prestigious, and famous, mathematicians alive today. The details of the study and the results are treated in chapter six.

Journaling: Preservice Teachers

Clearly, things were going well. I had recovered from my earlier failures and was now sitting with data from two successful surveys. But, I was not satisfied. The main reason for this was that from the moment I had decided to use anecdotal accounts for my data I had been requiring that the students in my *Designs for Learning Mathematics* courses were to keep problem solving journals. The reason for this was not entirely selfish, as I saw great merit in its use as both a learning tool and an assessment tool. However, what bothered me was that for two years I had been completely unsuccessful in getting the students to write in the journals in a manner that was truly reflective of their problem solving process. I felt that if I could somehow manage to make this happen that the richness of the data that they would hold could surpass what I had thus far. The problem was, their journaling with regards to this was poor.

The students were too set in their mathematical ways. They would write their journals as if they were writing a solution; *first I did this … then I tried that … etc*. I knew from working with them on their problems that the way they wrote about their solution was not at all reflective of what really occurred. Their actual problem solving experiences were fraught with failed attempts, wrong turns, and progress that moved in fits and jerks, oscillating between periods of inactivity, stalled progress, rapid advancement, and epiphanies. However, their writing did not reflect these experiences. When I brought this to their attention they evolved their journaling to include some of their failed attempts and frustrations, but for the most part it remained a narration of their mathematical solution and not their mathematical process. I wasn't going to find any sincere data in such reflections. If I was to use their journals as a source of anecdotal data I had to find a way to improve their journaling ability.

Then for no reason other than sheer luck I was given a copy of a book chapter written by Douglas Hofstadter (1996). In this chapter Hofstadter tells the story of a mathematical exploration that he had engaged in which eventually resulted in an AHA! experience. This was by far the best accounting of a mathematical problem solving process that I had ever read, and the rich description of the AHA! aside, it provided me with a template for effective journal writing. I dissected the format of the writing and turned this description into a prescription for effective accounting of true problem solving accounting that I could teach to my students. I changed the way I approached journaling with my students and taught them according to the new framework I had developed. The results were pleasing. The intricacies of this framework, as well as the details of the study and its ensuing results are presented in chapter seven.

CHAPTER FIVE
AHA! AND AFFECT: THE CASE OF UNDERGRADUATE MATHEMATICS STUDENTS

As outlined in the previous chapter, the study presented here was my first attempt at collecting anecdotal data for the purposes of answering my research questions. The anecdotal evidence that I eventually did collect was in the form of reflective anecdotal accounts of, for the most part, recent AHA! experiences by students enrolled in an undergraduate mathematics course. Details about the course and the participants follow in the next section of the chapter. However, before moving on to these details I want to make a brief, and partial comment on the transformative nature of this particular study.

Although not reflected in the research questions presented in chapter three, I began this journey of unlocking the essence of the AHA! experience firmly believing, and intending to show, that the AHA! is rooted in the magnitude of the mathematical understanding that is achieved at the moment of illumination. This very cognitive view of the AHA! experience was not unfounded. As discussed in chapter one there is a rich basis in the literature for such an assumption to rest on. Both Hadamard (1945) and Poincaré (1952) discussed it, as did the numerous authors who presented on the AHA! in the context of creativity. I firmly believed that my research would follow on this path towards the cognitive dimension of the AHA! experience.

As very quickly becomes apparent, though, I did not stay on this path for very long. It was during my work on the study that I present in this chapter that I was alerted to the possibility of pursuing the AHA! experience from the perspective of the affective dimension. As a result, my views on the AHA! experience were dramatically and permanently changed. In what follows I present this study in the truest form that I can, preserving for the reader the evolution of my thinking.

Methodology

In the context of a course project I had decided to administer what can best be described as a survey. As detailed in chapter one and recapitulated in chapter four, Hadamard (1945) had had great success in his use of his survey, albeit that he used it to solicit ideas from mathematicians and not students. In addition, Mary Barnes (2000) had had great success in collecting data from mathematics students, albeit she did so with the use of observations and interviews and not a survey. I chose to merge these two methodologies.

The Course, the Participants, and the Teaching Assistant

The participants for this study were undergraduate students at Simon Fraser University enrolled in a 'Foundations of Mathematics for Teachers' course (MATH 190). At the time of the study this course was one of a number of courses that could be taken to satisfy a mathematics prerequisite for entry in a teacher education program for prospective elementary school teachers. However, of all the options available, enrolment in MATH 190 was by far the most common route towards satisfying this requirement and as such was populated almost entirely by prospective teachers.

This course had been designed with the intention of providing its enrolees a foundational understanding of elementary school mathematics. There is a focus on conceptual understanding of topics as opposed to an ability to replicate procedural algorithms. There is an attempt to look at specific strands of mathematics such as geometry and number theory in their entirety as opposed to the piecewise and fragmented way in which mathematics is often experienced in a spiralled curriculum. There is also an attempt to integrate an underlying appreciation for mathematical thinking and reasoning across all strands of the course. To put it simply, MATH 190 is regarded as an 'unpacking' (or 'repacking') course. It lays out all of elementary mathematics at one time for examination and conceptual reorganization. The hope is that such an approach would allow the students to build connections between individual strands of mathematics and facilitate a deeper understanding of the topics as they make sense of all that they see. The course runs for the length of one semester (13 weeks) and has four contact hours each week. The course mark is determined from performance on weekly assignments (10 in total), a project, two midterms, and a final.

The course is also specifically delivered with the enrolees, themselves, in mind. In general, students enrolled in MATH 190 are best described as *resistant*. That is, they are resistant to the fact that they have to take a mathematics course. Many of the students would describe themselves as either being math-phobic, math-incapable, or a combination of the two. They usually have negative beliefs about their abilities to do mathematics, poor attitudes about the subject, and dread the thought of having to take a mathematics course. Such being the case, there is a great effort made to alleviate some of these anxieties in the pedagogical approaches to the course. This is more than an emphasis on making the course content seem less daunting, although such an emphasis is certainly done. For example, students are expected to complete their assignments in groups of three to five members. It is felt that in addition to group work being a positive method for learning it is also a structure for support and encouragement. Students are also provided with support through an open tutorial lab. The lab is open to students from 20 to 30 hours per week and is staffed at all times with between one and four teaching assistants. This is a place where students can go to seek help should they need it as well as a place to meet with their groups and work on assignments.

At the time of the study there were 112 students enrolled in the course and I was working as the teaching assistant assigned to them. My role in this regard consisted of marking all of the assignments and projects, participating in the invigilation and marking of the midterms and the final, and I was scheduled to work in the lab for three hours immediately after each lecture. As a result, I was significantly involved in the students' learning of the course material and well informed of their progress in this regard.

Creating the Survey

The genesis of the survey eventually used in this study was born out of informal discussions with students as they worked in the lab setting. Students were engaged in a discussion early on in the course regarding AHA! experiences in mathematics. At the time there was no explicit purpose to these interviews other than to use them as a way to enter into a dialogue with the students in an attempt to get to know them. Given that I had a deep interest in the AHA! experience the choice of this as the topic with which to engage them came about naturally. The discussions were very informal and can best be described as conversations.

The result of these conversations can be summarized as producing two very prominent trends. The first trend was that students, by and large, knew what was meant by an AHA! experience, even without definition. Many had alternate names for it referring to it as '*when it clicks*' or '*that spark*'. Some students assumed I was referring to '*when you suddenly remember the name of someone or something that was on the tip of your tongue, but you couldn't remember*'. Only a very few students needed me to clarify what I meant by an AHA! experience. However, I did provide clarification for everyone, if only to situate the phenomenon, and the discussion, in mathematics.

The second theme that emerged from the informal interviews was how few of the students could recall an AHA! experience in the context of mathematics, most of them claiming to never have experienced one. As the course progressed, however, I began to witness and hear about how more and more students were encountering this phenomenon in their work on the course content. I realized that this was an ideal population from which to collect anecdotal data regarding the AHA! experience.

This realization prompted the creation of the survey for the express purpose of capturing some of these experiences. As already mentioned, part of the course requirements obliged the students to produce an end-of-term project. Between the course instructor and myself it was agreed that one of the options for this project would be for them to write about an AHA! they had experienced in the context of mathematics during their participation in the course. As it was not certain that everyone could claim to have had an AHA! experience the students were offered an alternative to this assignment. They could, if they wished, engage in a mathematical investigation centred around a problem to solve. In order to be fair this option was open to all students regardless of their experience with an AHA!. The project was worth 10% of their final mark and they were given four weeks to work on it. For details of the project in its entirety see appendix D.

Because this was my first attempt at collecting anecdotal accounts of AHA! experiences I did not know what sort of data such an approach was capable of, or incapable of, producing. The literature had given me some idea, but not enough to say for sure how my particular participants would respond to the survey. I could not anticipate what I could or would discover, and I did not want to take the chance that I would miss something. Hence, I designed the survey to be as inclusive of the experience as possible. I decided to attempt to collect data on every aspect of the experience, from what the mathematical context of the experience was to how the experience made them feel. In short, I was interested in what sort of data it is possible to collect through anecdotal recounting. The final version of the survey appears below.

> "I had been working on the problem for a long time without any progress. Then suddenly I knew the solution, I understood, everything made sense. It seemed like it just CLICKED!"
>
> *The above anecdote is a testament of what is referred to as an AHA! experience. Have you ever experienced one? The purpose of this assignment is to have you reflect upon such an AHA! experience and to explore exactly what you learned in that instance and what you think contributed to the moment. You will hand in:*
>
> 1. *A detailed explanation of the specific mathematical topic that you were studying and the difficulty you were having with it (including any incorrect or incomplete understandings that you had of the topic before the AHA!).*
> 2. *The story of the AHA! experience as you remember it, paying particular close attention to what you were doing before it happened, when it happened, and how it made you feel when it happened.*
> 3. *A detailed explanation of your new understanding of the mathematical topic.*
> 4. *A conclusion as to how, upon reflection, the AHA! experience contributes to mathematical learning in general, and for you in particular.*
> 5. *Anything else that you feel would contribute to the reader gaining insight into the moment as you experienced it.*
>
> *Your final product will be evaluated for completeness and clarity.*

Rather than provide a definition of the AHA! experience, such as the one developed in chapter one, I chose instead to provide the participants with an anecdotal account using the language that the students had used in my informal interviews with them earlier in the semester. In order to provide the participants with a larger focus of the survey, I also prefaced the questions with a brief statement of the overall purpose of the assignment, in the hope that this would prevent them from getting overly caught up in the details of the individual questions. As can be seen from each of the five questions, I was leaving no foreseeable aspect of the AHA! experience unexplored. To access the initiation phase of the AHA! experience, question one focused specifically on the mathematical context within which the AHA! occurred. Question two was designed to elicit a response regarding the illumination phase of the experience and focused on the story of the AHA! as well as on the feelings invoked by it. Question three was the continuation of question one and was included in order to measure the change in understanding of a mathematical topic across the AHA! experience. The last two questions were included as a way to allow the participants to present their own ideas on the phenomenon should they have any.

Response Rates

Of the 112 students enrolled in the course, 76 students chose to write about their AHA! experience. Of these, 65 recounted such an experience in the context of their course experience. The remaining 11 participants responded to the survey in the context of an AHA! that they had had in their pre-tertiary schooling. I did not dismiss these accounts, but I did analyse them separately.

The 76 responses can be grouped into two categories: *teaching* and *discovering*. The first of these, teaching, pertains to AHA!'s experienced in the passive reception of mathematical content. In total, there were 14 accounts of such AHA! experiences. Each of these told of an event in the lecture hall where something the instructor said or demonstrated caused them to understand a previously not understood piece of mathematical content. A much more common AHA! experience, however, was of the discovery type and accounted for the remaining 62 responses. These were descriptions of AHA!'s that had occurred in the context of trying to work something out for themselves, either in solving a problem or working towards understanding some particular mathematics content. The 11 accounts of AHA! experience that fell outside of the context of the course were also distributed between these two types of AHA!'s with 6 in the teaching type and 5 in the discovery type. These results are summarized in Table 1 below.

This table clearly shows the predominance of discovery type AHA!'s over teaching type AHA!'s. This is not surprising given that to discover something presupposes an AHA! experience, whereas to learn something through the passive reception of being taught to does not.

Table 1: Undergraduate Mathematics Students' Response Rates by AHA! Context and AHA! Type

	Teaching AHA!'s		Discovery AHA!'s	
	number	% of n (n=76)	number	% of n (n=76)
AHA!'s within the course	8	11%	57	75%
AHA!'s prior to the course	6	8%	5	7%
Totals	14	18%	62	82%

An Overview of the Responses

As detailed in a previous section, prior to administering this survey I was not aware of what sorts of data such a survey was capable of producing. As can be seen from the questions I chose to include in the survey, I was hoping that detailed descriptions of the mathematical context of the AHA! experience would provide me with some insights into the stimuli that invoked the experience. I was also hopeful that descriptions of the newfound understandings of the participants after the AHA! experience, again situated in the particulars of the mathematical context, would provide insight into the cognitive transformation created by the experience. My first reading of the data was focused on this one aspect. I quickly realized that my hopes would be wholly, and completely, unrealized. The data presented account after account of mathematical AHA! experiences in which the cognitive transformation was unremarkable. I present here Kim's comments as an example.

> *My AHA! experience came during the first midterm. The question that I was having difficulty with was the very first one. That was already a bad sign. The question was concerning patterns and involved stars arranged in a triangular fashion.*

P(1)	** *
P(2)	*** ** *
P(3)	**** *** ** *
…	…

The question asked how many stars would be in P(12) and how many stars in P(121)? I was lost. I added up all the stars for each number and compared them to each other and tried to figure out the pattern there. I drew diagrams that made no sense. I tried to use formulas that I had memorized, placing n's and (n-1)'s and (n+1)'s anywhere, but that was not yielding anything correct. Everything I thought I remembered was completely disintegrating out of my brain and as the minutes ticked I began to panic. As I turned the page there was a similar question as the first. I was lost again. The exam is going very poorly. I ultimately decided to skim throughout the rest of the test and purely answer whatever I could and if I had moments to spare I would go back to the puzzling questions. I went back to question one and looked at what I had written down thus far. Then I erased it all. I started to draw more figures representative of the ones given ***and then it hits me!*** *I realized that the number of stars in the first row of any P(n) was one larger than n and that they descended by 1 to 1. that is P(4) starts with 5, then 4, 3, 2, 1. I now remember the very first math class where we discussed patterns and Gauss and how to add all the numbers together up to 100. So, P(12) = 13+12+11+…+1.*

> *13+12+11+…+ 1*
> *<u>1 + 2 + 3 +…+13</u>*
> *14+14+14+…+14 = 14x13 = 182*
> *So, P(12) = 182/2 = 91 (because we counted each twice)*

Thus, the answer is 91 stars for P(12). I could now answer the rest of the questions with ease. I began to relax …

In general, all these account showed that prior to the AHA! experience there existed in the mind of the participant a piece of mathematics that they either did not understand or a problem that they could not solve. For all intents and purposes they were 'stuck'. After the AHA! experience they now understood this mathematics or could solve the problem. They were 'unstuck'. The AHA! was clearly in the middle of all this and intimately involved in the transition from being 'stuck' to being 'unstuck'. However, at the level of mathematical understanding this transition from 'stuck' to 'unstuck' was minute. It was unremarkable and in many cases indistinguishable from simply having learned something. It was clear that there was an experience of some importance, but that importance was not played out at the level of mathematical understanding.

All was not lost, however, for as unremarkable the data was at a cognitive level, it was exceptionally remarkable at an affective level. This was completely unanticipated. Yes, I had created question two for the survey, and I had included a reference in this question as to how the AHA! experience made them feel. At the time of creating the question I was mostly interested in the details of what it was that they were doing immediately prior to their AHA! in an attempt to tease out if there was an form of discernable stimuli that precipitated the event. This inclusion of feelings had been incidental in an attempt to provide the participants a context with which to discuss their experience. Regardless of my intention, my reference to feelings was like a lightning rod for the participants. They responded specifically to this query and did so with conviction and passion. More importantly, there were clear trends in these responses that warranted a serious second look. In the time it took to do an initial reading of the data the focus of this study shifted from the cognitive domain to the affective domain.

The Affective Domain

The learning of mathematics has classically been studied from the perspective of cognition, that is, the examination of what are the cognitive processes involved in learning, and how they operate. However, the inability of such research to explain the failures of people in problem solving contexts who possess the cognitive resources necessary to succeed has prompted the re-evaluation of the role of the affective domain in the learning of mathematics (Di Martino & Zan, 2001).

The affective domain is most simply described as *feelings* – the feelings that students have about mathematics. In general it is understood that the affective domain is composed of three dimensions: beliefs, attitudes, and emotions (McLeod, 1992). The beliefs are, just that, what students believe; what they believe to be true about mathematics and what they believe about their ability to do mathematics. Beliefs about mathematics are often based on their own experiences with mathematics. For example, beliefs that mathematics is 'difficult', 'useless', 'all about one answer', 'all about problem solving', or 'all about memorizing formulas' stem from experiences that have first introduced these ideas and then reinforced them. Research has shown that such beliefs are slow to form in a learner, and once established, are equally slow to change, even in the face of intervention (Eynde, De Corte, & Verschaffel, 2001).

A qualitatively different form of belief is with regards to a person's beliefs in their ability to do mathematics, often referred to as efficacy, or self-efficacy. Self-efficacy, like the aforementioned belief structures, is a product of an individual's experiences with mathematics, and is likewise slow to form and difficult to change. Self-efficacy with regards to mathematics has most often been dealt with in the context of negative belief structures (Ponte, Matos, Guimarães, Cunha Leal, & Canavarro, 1992) such as 'I can't do math', 'I don't have a mathematical mind', or even 'girls aren't good at math'.

Attitudes can be defined as "a disposition to respond favourably or unfavourably to an object, person, institution, or event" (Ajzen, 1988, p. 4). Attitudes can be thought of as the responses that students have to their belief structures. That is, attitudes are the manifestations of beliefs. For example, beliefs such as 'math is difficult', 'math is useless', or 'I can't do math' may result in an attitude such as 'math sucks'. A belief that 'math is all about formulas' may manifest itself as an attitude of disregard for explanations in anticipation of the eventual presentation of a formula. Attitudes, like beliefs, are stable entities, they are slow to form and difficult to change.

Emotions, on the other hand, are relatively unstable (Eynde, De Corte, & Verschaffel, 2001) and, as a result, the role that they play in the learning of mathematics has received little attention (for exception see DeBellis & Goldin, 1999, 1993). They are rooted more in the immediacy of a situation or a task and as a result are often fleeting. Students with generally negative beliefs and attitudes can experience moments of positive emotions about a task at hand or, conversely, students with generally positive outlooks can experience negative emotions. Changes in beliefs and attitudes are generally achieved through the emotional dimension, repeated negative experiences will eventually produce negative beliefs and attitudes, and likewise, repeated positive experiences will produce positive beliefs and attitudes. However, as mentioned earlier, existing literature indicates that change is slow.

Because of their stable nature, research has focused primarily on the role of beliefs and attitudes on the learning of mathematics (Eynde, De Corte, & Verschaffel, 2001). The results indicate that beliefs and the resulting attitudes are strongly linked to school achievement (Leder, 1992; Ponte, Matos, Guimarães, Cunha, Leal, & Canavarro, 1992). They are the gatekeepers to learning. Before a student can even begin to engage in mathematical content they have to first decide that they are both capable of learning the presented material, and willing to do so. Once this has been decided any residual beliefs and attitudes regarding the learning of the content will continue to affect their learning of it. For example, a student with the belief that mathematics is about 'memorizing rules' will approach new content matter from the perspective of identifying a rule, mastering the use of that rule, and then memorizing that rule – regardless of the intended cognitive outcome of the lesson.

Given the change in focus from the cognitive domain to the affective domain, the data was recoded and reanalysed according to the theories regarding affect present in the literature, with specific focus on emotions, beliefs, and attitudes, as well as on the changes in beliefs in attitudes. The results of this reanalysis are presented below.

Participants' Responses

> *In that moment when the connection is made, in that synaptic spasm of completion when the thought drives through the red fuse, is our keenest pleasure.*
>
> - Harris (2000, p. 132)

As already mentioned 76 students chose to write on their AHA! experience. Of these, all but one mentioned how they felt when they experienced the AHA!, even if only to say 'I felt great'.

However, not all of the students portrayed an accurate understanding of what was meant by an AHA! experience. Although there were no false AHA!'s – AHA! experiences that proved to be unfounded as discussed in chapter one – there were three cases of misunderstood AHA!'s. One student took the assignment to mean that she was to construct an AHA! experience. I know this because she came into the lab and sat down in front of me and tried to create an understanding of Venn diagrams, a topic that until then had troubled her. In fact, through my tutelage she did manage to construct an understanding of Venn diagrams and then, unbeknownst to me, proceeded to write about this process for her assignment. This would have been fine if there had, indeed, been a moment of illumination within the process, but there hadn't. Instead, there was an observable slow awakening to the concept. Evidence of thinking of the AHA! experience as a slow dawning of understanding was also present in the writing of two other students. Andrea even went so far as to rename it the *AAAHA! experience*.

> *Andrea:* *My AHA! came slowly – not all at once, but little by little I grasped the concept.*

The remaining 73 students all presented experiences that were consistent with what was expected in the spirit of the assignment (i.e. true AHA! experiences). These responses were recursively coded according to affective themes that were emerging from the data. In what follows I present these themes through the discussion of representative excerpts from students' responses. That is, for each theme I use excerpts that exemplify the themes while at the same time being representative of all of the students' responses pertaining to that theme. These excerpts stand alone, away from the mathematical context in which they occurred. I do this for two reasons. The first is that, as discussed above, my initial analysis of the data showed that the mathematical context and the progression of understanding within this context was both unremarkable and unhelpful in the consideration of my research questions. The second reason is that by presenting these excerpts in their decontextualized form the themes are much more apparent.

Anxiety

Although the topic of anxiety was not brought up in the context of the AHA! experience, 34 students felt it necessary to mention how they felt about mathematics, or about taking a mathematics course. They seemed to do this as a way to provide a baseline for a discussion of their changing feelings. In these 'baseline' discussions the theme of anxiety was prevalent, manifesting itself in terms of dislike, fear, apprehension, and traumatic memories.

Jennifer, Stephanie, and Tonia reflect on how they feel about having to take this course in order to be able to enter into the teaching program. While Jennifer states a dislike for mathematics, Stephanie and Tonia express a fear of the subject matter.

> *Jennifer:* *I have never been a person that likes or even enjoys math at all, so the idea of having to take this class if I wanted to teach wasn't very appealing to me. So I came into the course with the preconception that it would be just like any other math class that I had taken.*
>
> *Stephanie:* *When I entered MATH 190, I felt that fear in my stomach return. I needed this course to enter teaching so the pressure was on.*
>
> *Tonia:* *I was scared of the subject as a student and this was magnified 100 times as a teacher. I knew I had to take this course because I did not want my students to feel the same way as I did about math.*

Marcie reflects on her experience in mathematics in general – going all the way back to her negative elementary school experiences.

> *Marcie:* *I was feeling emotions that should not have even existed in grade school.*

As mentioned earlier, most students are resistant to taking this course. This resistance is by and large due to the anxiety they have towards mathematics. From the informal discussions I had with the students at the beginning of the course I learned that this anxiety, and resulting resistance, was so strong that many of them had deferred taking this course until their last semester of undergraduate studies. Given that this course was a prerequisite for entry into the teaching program, this 'waiting till the last minute' strategy created a new type of anxiety pertaining to the pressure of having to succeed, and succeed now. This is reflected in Maggie's comment.

> *Maggie:* *I never liked math when I was in school and so I had avoided it when I got to university. I knew I had to have this course in order to apply for PDP* [the teacher education program]*, but I put it off and put it off until now. It is the only course left for me to take before I can apply for PDP.*

Pleasure

All but one of the participants in this study mentioned something about how the AHA! experience made them feel. Although their comments varied in length and details with regard to these feelings, each of them stated in one way or another that it felt 'great'. In what follows I provide a partial list of some of these comments.

> *John:* *It felt great.*
>
> *Ruth:* *I was so relieved; I could barely contain my happiness.*
>
> *Jenny:* *This was the best feeling.*
>
> *Christina:* *I never knew I could feel so good while doing math.*
>
> *Keri:* *Wow!*
>
> *Stacy:* *The joy I felt was like none other.*
>
> *Natalie:* *It made me feel like I could do anything.*

It is clear that the AHA! experience produced a positive affective response in the students. However, as will be shown in the next two sections, the AHA! produced more than simply a 'good' feeling. It contributed to a positive change in the beliefs and attitudes of many of the students.

Change in Beliefs

Of the 76 students who chose to do their project on their AHA! experience 61 of them discussed their beliefs. Moreover, each of these 61 students did so in the context of changing beliefs. That is, they expressed a change in their beliefs through the experience of the AHA!

Susan describes how the experience has changed her beliefs in both her ability to solve problems and the process she uses to produce a solution.

> *Susan:* *The AHA! experience is inspiring. It makes students believe that they solved that question through reasoning and deep thought, and inspires him or her to seek more of these moments to obtain a sort of confidence and further knowledge.*

This was a common theme, often manifesting itself in discussions of newfound confidence as expressed by Steve and Andrea.

> *Steve:* *Initially this course made me very unsure of myself but now I am confident when working out problems among my homework group. Previously, I naturally deferred to them, but after this AHA! experience I got confidence in my answers.*

> *Andrea:* *In reflecting upon this AHA! experience I feel a sense of pride that I accomplished this mathematical idea by myself. I am relieved to know that I do not have to depend on others to help me along. This moment also gave me a self-confidence boost in the sense that I may have something to contribute to others, for example my group members.*

James reflects on how the absence of these experiences may have contributed to his belief that he was not good at mathematics.

> *James:* *For myself, I wish that I'd had more of these moments in my earlier years of high school then I would maybe not have so readily decided that I was not good at math.*

The belief of what 'it takes' to be good at math is altered for Lena as she expresses that she now sees that it is not an issue of intelligence.

> *Lena:* *Knowing that I could stare at a problem and in time I would understand, gave me more confidence that I could be successful in math. It really is not an intelligence issue.*

Karen sits on the border between beliefs in her ability to do mathematics and her belief in what it takes to do mathematics.

> *Karen:* *I used to think that if you couldn't get it right away you didn't know how to do it. This is the longest I've ever worked on a problem. I had just about given up when it just came to me. I now know that sometimes it just takes time.*

Although Karen's response was similar to that of one other student, her response is unique in that she arrived at this conclusion in the context of doing the other option for the final project. Karen had not intended to write on an AHA! experience and so chose to pursue the problem solving option of the assignment. It was during her work on this problem that she had, what she claims to be, her first mathematical AHA! experience.

What is interesting is the variety of beliefs that were affected by experiencing illumination in the context of mathematics. Although most of them centre on their own conceptions of their abilities to do mathematics some students expressed how their beliefs about mathematics have changed, as seen in Paula's statement.

> *Paula:* *I used to think that math was all about the right answer, but now I am more aware of the value of the process.*

Change in Attitudes

Because attitudes are the manifestations of beliefs it was sometimes difficult to distinguish between the two. That is, almost every expression of a change in attitude had a discernable change in beliefs associated with it – and has been counted in the 61 responses discussed above. Charlotte and Stephen express a change in optimism and expectations, respectively.

> *Charlotte:* *I have a better attitude now; I'm more optimistic. This is helpful in learning as complete thought processes can be impeded by a dejected attitude.*

> *Stephen:* *Also, I enjoy math now. I feel like this success stimulated more success. Now I have raised my expectations in math.*

Carla has come to terms with her lack of knowledge of mathematics and found within it a new attitude for success.

> *Carla:* *I've decided that I really don't know a lot of math. But who cares? I know enough. And I know how to think enough to find the answers. And I know how to ask for help. And I don't care so much about the end result.*

However, a few students clearly demonstrate a change in attitude without expressing an obvious change in beliefs. This is best demonstrated in Kristie's comment.

Kristie: *I must admit that math is challenging for me … after the AHA! experience you feel like learning more, because the joy of obtaining the answer is so exhilarating. It almost refreshes one's mind and makes them want to persist and discover more answers. It gave me the inspiration and the determination to do the best that I can do in the subject.*

Kristie has most definitely changed her attitude about the pursuit of mathematics in that she is feeling inspired and determined to succeed in the course. What is not clear is whether or not this is as a result of a new belief that she can succeed.

Analysis

Almost all of the participants alluded to a sense of accomplishment that accompanies the AHA! experience, most actually using the word 'accomplishment' to describe the feeling. However, it should be noted that this sense of accomplishment is a secondary result of the AHA! experience, the primary result being the successful solution of a problem or the coming to understand a piece of mathematics. That is to say, a sense of accomplishment comes from accomplishing something. Deanna demonstrates this nicely.

Deanna: *After I understood the question and I had completed it, I felt as though I had accomplished something. I felt as though I was somewhat complete in my understanding of the problem.*

I make this distinction for one very important reason, to contrast the effect that accomplishment has on beliefs and attitudes with the effect that the AHA! experience has.

There is a wealth of research that indicates that success and feelings of accomplishment contribute to a change in attitudes and beliefs (Leder, 1992; Ponte, Matos, Guimarães, Cunha, Leal, & Canavarro, 1992). However, the change they produce is minute. Long periods of sustained and successive success are required to create significant change. This is why beliefs and attitudes are considered to be stable in nature and why positive experiences are claimed to be so important in teaching and learning of mathematics. However, I question this view. The data in this study clearly shows that beliefs and attitudes can be drastically changed through a single AHA! experience. This is not the mark of stability. It may be true that these dimensions of the affective domain resist change in the face of successful completion of mathematical activity, but they yield easily to the phenomena of the AHA! experience. The question remains, however, by what mechanism is such drastic change in the affective domain possible?

I have two possible explanations for this phenomenon. The first is that the positive emotion that is achieved during an AHA! experience is much more powerful than the emotions that are achieved through non-illuminated problem solving. As a result, the effect that they have on beliefs and attitudes is that much more drastic. Furthermore, an AHA! experience often presupposes an accomplishment. Perhaps the sense of accomplishment is heightened and intensified through the mechanism of discovery, once again producing that much more change in the affective elements of beliefs and attitudes.

The second explanation has to do with inspiration. Having solved something challenging, or understood something difficult, besides being a great accomplishment is also a measure of what is possible. Success breeds success, and the students seemed to know this. They were inspired to continue, to get better.

> *Elizabeth:* *AHA moments are those great moments of deeper understanding and clarification of problems where incorrect or incomplete understanding is overcome. These moments inspire us and encourage us to keep going despite the frustration and anxiety that often tends to overwhelm us in times of difficulty when attempting to solve a problem.*

Elizabeth articulates very nicely that AHA! experiences 'inspire us and encourage us to keep going'. With such motivation success seems to be inevitable. Perhaps it is the anticipation of greater mathematical understanding and ability that changes beliefs and attitudes for the future. This is exemplified in David's optimistic outlook on mathematics and the AHA! experience.

> *David:* *The moment of comprehension is what keeps 'wannabe' mathematicians in the game. The hope that one day, in one instant, the world will mysteriously come into alignment and math will make sense.*

Conclusion

To do empirical research in the social sciences one draws from a substantive domain, a conceptual domain, and a methodological domain. The substantive domain provides a set of real world phenomena, the nature of which can be pondered. The conceptual domain allows for the situating of ideas about these substantive phenomena in the theories of the field and to push them against each other in order to formulate a set of concise and pointed research questions. Finally, the methodological domain provides the means to design studies that answer the research questions. In this study I drew from each of these three domains. I had the very real, and well-documented phenomenon of the AHA! experience. I drew on the theories in the literature regarding the significance of the mathematical understanding that is achieved during such experiences. Finally, I used prior research methodologies pertaining to the AHA! experience as templates for the design and implementation of this study. Having done all this I had every expectation that my results would speak to my research question. This did not happen. I set about looking for evidence and answers regarding the AHA! experience in the cognitive dimension of the students' experiences, but instead I found evidence and answers regarding the AHA! experience in the affective dimension of the students' experiences.

As much as the occurrence of the AHA! experience may be situated in the cognitive enterprises of problem solving and coming to understand, it actually manifests itself in the affective domain. The positive emotions that it invokes has the power to change negative beliefs and attitudes about one's ability to do mathematics as well as negative beliefs and attitudes about the subject of mathematics itself. For these reasons, the impact that an AHA! experience can have on students learning is not to be ignored. Through the restructuring of the affective domain we can endeavour to feed into their cognitive processes more directly. That is, we can get past the gatekeeper that is the affect and begin to access student learning.

I found something I wasn't looking for, and I found it where I wasn't looking. This turn of events completely changed my way of seeing and thinking about the AHA! experience and as a result allowed me to extend my pursuit of the experience in new and meaningful ways. It guided my interpretation of the data I present in chapter six and it helped to formulate the study I detail in chapter seven.

CHAPTER SIX
AHA! EXPERIENCES: THE CASE OF PROMINENT MATHEMATICIANS

In 1902, the first half of what eventually come to be a 30 question survey was published in the pages of L'Enseignement Mathematique, the journal of the French Mathematical Society. Édouard Claparède and Théodore Flournoy, two French psychologists, who were deeply interested in the topic of mathematical creativity, authored the survey. Their hope was that a widespread appeal to mathematicians at large would incite enough responses for them to begin to formulate some conclusions about this topic. The first half of the survey centred on the reasons for becoming a mathematician (family history, educational influences, social environment, etc.), attitudes about everyday life, and hobbies. This was eventually followed up, in 1904, by the publication of the second half of the survey pertaining, in particular, to mental images during periods of creative work. The responses were sorted according to nationality and published in 1908, but for reasons that will soon become clear, quickly faded into obscurity.

By the time that Claparède and Flournoy's survey was published, one of the most noteworthy mathematicians of the time, Henri Poincaré had already laid much of the groundwork for his own pursuit of this same topic. Consequently, he did not respond to the request published in L'Enseignement Mathematique, and for reasons that are not clear, neither did many of his peers. In fact, of those mathematicians that did respond to Claparède and Flournoy's survey, none could be called noteworthy (Hadamard, 1945). What was more damaging, however, was the fact that shortly after Claparède and Flournoy published their results, Poincaré gave a talk to the Psychological Society in Paris entitled Mathematical Discovery. At the time of the talk Poincaré stated that he was aware of Claparède and Flournoy's work, as well as their results, but stated that they would only confirm his own findings (Poincaré, 1952).

Another noted mathematician who did not respond to Claparède and Flournoy's survey was Poincaré's good friend Jacques Hadamard who would turn out to be the biggest critic of Claparède and Flournoy's work. Hadamard felt that the two psychologists had failed to adequately treat the topic of mathematical creation on two fronts; the first was the lack of comprehensive treatment of certain topics and the second was the lack of prominence on the part of the respondents. Further, Hadamard felt that as exhaustive as the survey appeared to be, it failed to ask some key questions – the most important of which was with regard to the reason for failures in the creation of mathematics. This seemingly innocuous oversight, however, led directly to what he termed "the most important criticism which can be formulated against such inquiries" (1945, p.10). This leads to Hadamard's second, and perhaps more damning, criticism. He felt that only "first-rate men would dare to speak of" (p.10) such failures, and so, inspired by his good friend Poincaré's treatment of the subject Hadamard retooled the survey and gave it to friends of his for consideration –

mathematicians such as Henri Poincaré and Albert Einstein, to name a few, whose prominence were beyond reproach. Ironically, the new survey did not contain any questions which explicitly dealt with failure. In 1943 he gave a series of lectures on mathematical invention at the École Libre des Hautes Etudes in New York City. These talks were subsequently published as *The Psychology of Mathematical Invention in the Mathematical Field* (Hadamard,1945).

Hadamard's treatment of the subject of invention at the crossroads of mathematics and psychology was an entertaining, and sometimes humorous, look at the eccentric nature of mathematicians and their ritualistic practices. His work is an extensive exploration and extended argument for the existence of unconscious mental processes. So, when looking to collect anecdotal accounts of AHA! experiences from prominent mathematicians, I chose to use Hadamard's work as a template for my methodology.

Hadamard's Survey Resurrected

How do you get a Fields Medallist to respond to a survey asking about their AHA! experiences? This was the question posed to me by Peter Borwein when I told him of my desire to collect reflective anecdotal data from prominent mathematicians. Clearly the medium of choice was email, but as many issues that this solved, it presented even more. The people I was interested in surveying received hundreds of emails every day – how was I to make mine stand out sufficiently to first attract, and then keep, their attention?

An equal, but qualitatively different, challenge was trying to get these mathematicians to respond in such a way as to provide me with the greatest amount of detail and sincerity. I knew from casual conversations with a number of mathematicians that the topic I was pursuing was an inherently interesting one. So, I knew that a direct question regarding illumination and/or insight would furnish good responses. However, I wanted more than this. I wanted to see how they perceived their thinking process in the act of creating and exploring new mathematics.

These were professional mathematicians and, as such, their language of trade was deductive logic and formal proof. They wrote, spoke, and taught in this language. But did they think in this language? In part, they did, but I was interested in the part where they did not. I was interested in their thinking process around their AHA! experiences. I suspected that direct questioning with regards to this would not provide me with the most sincere or complete answers. I decided, instead, to break my survey, as Claparède and Flournoy had done, into two parts. The first part of the survey would comprise a series of questions that would deal with the topic more tangentially, and would be sent out to a large number of prominent mathematicians. The second part of the survey would deal with the AHA! more directly and would be sent only to those who responded to the first survey. The question still remained, however, as to what these questions would look like.

The answer to all of these hurdles lay in Hadamard's original survey. Not only could he provide me with a template for methodology, he could also provide me with a title and the research questions[11]. Jacques Hadamard and his book *The Psychology of Invention in the Mathematical Field* were familiar to most, if not all, contemporary mathematicians. By couching my own work in the context of Hadamard's seminal work I greatly increased my chances for success. So, the first survey was designed entirely around the presumption that I was resurrecting part of Hadamard's original survey and given the title *Mathematics and the Psychology of Invention*.

Constructing the First Survey

Hadamard's original survey consisted of 33 questions (see appendix E) and had been given to friends of his, people whom he respected, and that he felt had something to contribute. These people, in turn, respected Hadamard and felt he was doing interesting work. So, they responded to the survey. Other than doing interesting work, I shared none of these other advantages with Hadamard. Based on advice from Peter Borwein I knew that if I wanted any sort of a response rate I had to be very careful in the construction of this first survey. I had to reduce the size of the survey drastically, write a cover letter that was flattering, informative, and attention grabbing, and I had to pack it all into an email that was no larger than two pages. This was no small task and with Peter's help I spent over a

[11] The questions that were eventually chosen, although from Hadamard's survey, were originally included in Claparède and Flournoy's survey.

month working on it. I opted for a newspaper style of writing where the attention grabber is out front and the details are at the back. I started with the sentences:

> *I am a doctoral student working on my dissertation in mathematics education and I am asking for your help. I am attempting to reproduce part of Hadamard's classic survey (see below). The particular questions to which I'm most interested in getting answers are:*

This was immediately followed by the five questions I had chosen and a brief discussion of the history of Hadamard's survey. I also included a copy of Hadamard's survey as an attachment (see appendix E) should they prefer to respond to it directly, either in part or in whole.

Choosing the questions that I wanted to include was probably the most difficult part of constructing the survey. I wanted to stay true to Hadamard's survey, while at the same time reducing the number of questions from 33 down to five. I began by selecting those questions I felt were most pertinent to my research, and then began a process of elimination. In the end I settled on questions 4, 6, 7, 9, and 16 from Hadamard's survey, sorted these into what I felt was an order of importance: 9, 7, 4, 16, 6. I also updated the language of the questions somewhat (see appendix F for complete final survey sent out). In the end, the questions that appeared in the survey were:

1. *Would you say that your principle discoveries have been the result of deliberate endeavour in a definite direction, or have they arisen, so to speak, spontaneously? Have you a specific anecdote of a moment of insight/inspiration/illumination that would demonstrate this? [Hadamard # 9]*

2. *How much of mathematical creation do you attribute to chance, insight, inspiration, or illumination? Have you come to rely on this in any way? [Hadamard# 7]*

3. *Could you comment on the differences in the manner in which you work when you are trying to assimilate the results of others (learning mathematics) as compared to when you are indulging in personal research (creating mathematics)? [Hadamard # 4]*

4. *Have your methods of learning and creating mathematics changed since you were a student? How so? [Hadamard # 16]*

5. *Among your greatest works have you ever attempted to discern the origin of the ideas that lead you to your discoveries? Could you comment on the creative processes that led you to your discoveries? [Hadamard # 6]* (Hadamard, 1945, pp. 137-141)

The first question, to me was the most important one. It embodied what I felt was the essence of the AHA! experience, and potentially the most fruitful direction of pursuit with research mathematicians. It pitted the formalist and logic notion of deductive reasoning against the more creative and extra-logical process of illumination (spontaneous arising of ideas). As such, it provided the mathematicians with both a language and a context with which to discuss their discoveries.

The second part of question one as well as question two were tailored to fit my purposes. I added the words *illumination* and *insight*. I debated the wisdom of this choice for a long time. I was worried that the explicit mention of illumination, in particular, spoke too directly at the AHA! experience. However, I wanted to provide the descriptors within the question in order to provide the language for some potentially interesting responses, so I decided to keep it. The third and fourth questions were chosen because they, like question one, provided potentially dichotomous contexts within which to discuss their mathematical activities. In particular I was interested in seeing how their work in creating mathematics contrasted with their work in assimilating mathematics, and if this had evolved with time.

The last question was, like question two, a sensitive one in that it spoke a little too directly at the topic. However, there was no other question in Hadamard's survey that could substitute for it and I needed for them to comment on their creative process. My hope was that some of the mathematicians, at least, had devised rituals that they followed in trying to insight moments of invention and invoke AHA!'s.

Having discussed the questions that I chose to include in the survey, I think it is prudent to also discuss two questions, in particular, that I chose not to include in my survey. They were:

> *Have you noticed that, occasionally, discoveries or solutions on a subject entirely foreign to the one you are dealing with occur to you and that these relate to previous unsuccessful research efforts of yours? [Hadamard #8]*
>
> *Have you ever worked in your sleep or have you found in dreams the answers to problems? Or, when you waken in the morning, do solutions which you had vainly sought the night before, or even days before, or quite unexpected discoveries, present themselves ready-made to your mind? [Hadamard #10]* (Hadamard, 1945, pp. 137-141)

These questions were originally selected as possibilities but were eventually excluded for two reasons. The first, as already discussed, was due to a need to keep the number of questions asked to a minimum. The second, and more pressing, reason was that besides speaking too directly at the AHA! experience, they asked for details that I hoped would emerge naturally from the respondents' discussions. By asking for these details directly the sincerity of these comments might have been compromised somewhat. It is for this reason also, that I hesitated to include question two and question five (above). However, in those cases I chose to proceed anyway either because I felt that the detail was sufficiently well insulated (question two) or sufficiently important (question five).

Selecting the Recipients

First of all, I think it would be prudent for me to comment on how I have been referring to the 'participants' of this study thus far. I have intentionally avoided the use of the word 'participants' because I feel that only those who responded to the survey participated in it. When discussing the intended audience of the survey, at large, I have referred to them as mathematicians, or prominent mathematicians. However, the time has now come to narrow that general audience down to the specific 'recipients' of the survey, and so I now add to my descriptors of the 'participants' the word 'recipient'.

Hadamard set excellence in the field of mathematics as a criterion for participation in his study. In keeping with Hadamard's standards I chose to do the same. I decided that I would survey only the most prominent of mathematicians. The question was how to judge prominence? More importantly, how to judge prominence in such a way that it would allow one to seek out participants? Luckily I did not need to answer either of these questions. Mathematicians have more than enough ways in which they recognize achievement in their field, from awards to societies. So, I decided to focus on the most prestigious of these awards, the Fields Medal[12] and the Nevanlinna Prize[13], and three academies, the American Society of Arts & Sciences[14], the Royal Society[15], and the Academie des Sciences[16] as recognized identifiers of prominent mathematicians. In the end I came up with a list of 150 names. These became the recipients of the email survey. This list included 30 Fields Medal winners, 5 Nevanlinna Prize winners, and 115 members of the three academies. To add sincerity to my inquiry there was no mass mail out. Each email was sent out separately, individually addressing the intended recipient by name.

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[12] "At the 1924 International Congress of Mathematicians in Toronto, a resolution was adopted that at each ICM, two gold medals should be awarded to recognize outstanding mathematical achievement. Professor J. C. Fields, a Canadian mathematician who was Secretary of the 1924 Congress, later donated funds establishing the medals, which were named in his honour (*The Fields Medal*). Consistent with Fields' wish that the awards recognize both existing work and the promise of future achievement, it was agreed to restrict the medals to mathematicians not over forty at the year of the Congress. In 1966 it was agreed that, in light of the great expansion of mathematical research, up to four medals could be awarded at each Congress." (European Mathematical Society, 2003)

[13] "The *Rolf Nevanlinna Prize* in mathematical aspects of information science was established by the Executive Committee of the International Mathematical Union (IMU) in April 1981. It was decided that the prize should consist of a gold medal and a cash prize similar to the ones associated with the Fields Medal and that one prize should be given at each International Congress of Mathematicians. One year later, in April 1982, the IMU accepted the offer by the University of Helsinki to finance the prize. The prize was named the Rolf Nevanlinna Prize in honour of Rolf Nevanlinna (1895-1980), who had been Rector of the University of Helsinki and President of the IMU and who in the 1950s had taken the initiative to the computer organization at Finnish universities." (European Mathematical Society, 2003)

[14] "Membership in the *American Society of Arts & Sciences* is by election only and is limited to men and women of the international community who demonstrate exceptional achievement, drawn from science, scholarship, business, public affairs, and the arts" (American Society of Arts & Sciences, 2003).

[15] "Each year 42 new Fellows and up to 6 new Foreign Members are elected from the most distinguished scientists. Election to the Fellowship of the Royal Society is recognised worldwide as a sign of the highest regard in science" (Royal Society, 2004).

[16] "The Academy of Sciences of the Institute of France brings together French scholars and forms associations with foreign scholars, where both the former and the latter are selected from among the most eminent. By their involvement, they contribute to the accomplishing of the missions of the Academy" (Academie des Sciences, 2004).

Response Rates

The first response came within one hour of being sent. This fact in no way contributes to my research other than it so filled me with satisfaction that I feel it necessary to mention it here. This response came form Enrico Bombieri, a Fields Medal winner in 1974, and was, in the end, the most prolific and detailed of all the responses.

Table 2: Prominent Mathematicians Response Rates by Source

Source	Recipients Selected	Responses	Response Rate
Fields Medalists	30	5	17%
Nevanlinna Prize Winners	5	0	0%
Academies	115	20	17%
Totals	150	25	17%

Within the first four weeks, there were 25 replies. Four of these replies acknowledged that, while they found my research to be of interest, they were far too busy to respond. Four weeks after the first email was sent I sent a reminder out to all those recipients that had not replied at all. This produced five more replies over the next two weeks. Two of these were to say that they were too busy to respond and one was to say that I should try them again in two months time, which I did – and received a response. Finally, 14 weeks after the first mail out I received a response that began, "It is time to answer your questionnaire", which I found to be delightfully to the point. In the end, I received 31 replies, 25 of which were responses to my survey and 6 of which were merely replying to say that they were too busy to respond. The responses are summarized in Table 2 (above) according to the source from which I found the name of the recipient.

THE SECOND SURVEY

As mentioned above, the respondents to the first survey were sent a second survey at a later date. This second survey consisted of two questions that were designed to solicit some very direct responses with regards to the AHA! experience. I present them here, along with the brief introduction that preceded the actual questions.

> *Your comments resonate with (or allude to) the idea of an AHA! or EUREKA! experience. I was wondering if you could comment a little bit more about such AHA! experiences? In particular, I would greatly appreciate it if you could answer the following two questions:*
>
> 6. *How do you know that you have had an AHA! experience? That is, what qualities and elements about the experience set it apart from other experiences.*
>
> 7. *What qualities and elements of the AHA! experience serve to regulate the intensity of the experience? This is assuming that you have had more than one such experience and they have been of different intensities.*

As can be seen by the explicit nature of the questions, their inclusion in the first survey could certainly have compromised the way in which the participants responded. Note that I made no attempt to define the AHA! experience for the participants. This omission was intentional. I had every confidence that the participants would know what was being asked without any elaboration.

Furthermore, I wanted to ensure that their responses were reflective of their own thoughts on the subject, rather than tailored to fit any definition I may provide.

This second survey was sent out as an email reply to their response to the first survey. This was done for two reasons. The first was to establish a conversational rapport with the participants that was more personal than simply sending out a second survey. The second reason was that this allowed me to remind the participants of how they responded to the first set of questions in a genuine and casual manner. Of the 25 mathematicians who were sent the second survey, 14 re-responded, with additional comments.

RESPONSES

The recipients who responded to the survey, in whole or in part, have come to be referred to as the 'participants' in this study. Their responses varied in length from a few short lines to eloquent four page letters. Some of them chose to speak directly to the questions, others chose to address the questions in the context of a more free-flowing commentary on mathematical discovery. Two participants chose to respond to Hadamard's survey that had been sent as an attachment to my survey.

The responses were initially sorted according to the survey question they were most closely addressing. This was difficult with those who formulated their response in more of an open format, or responded to the whole of Hadamard's survey, causing many of their comments to be placed in more than one question response. It also produced a number of 'leftovers'; comments that could not be associated with any of the survey questions.

A second sorting of the data was done according to trends that emerged in the participants' responses, regardless of which question they were in response to. This was a much more intensive and involved sorting of the data in an iterative process of identifying themes, coding for themes, identifying more themes, recoding for the new themes, and so on. In the end, however, this look at emerging themes took care of much of the leftovers, but more importantly, allowed for a qualitatively different reading of the data. In what follows I first discuss the answers to the seven questions from the first and second survey. I then discuss some additional themes that emerged. In all the discussions I use excerpts from the data as exemplars of the type of responses I received. I have chosen to cite these excerpts with the names of the participants[17], as I believe that who they are lends credence to what they say. Furthermore, I have often chosen to include more citations than are necessary for establishing my point for discussion. I do this for the reason that I find these anecdotal comments to be very interesting and pleasurable to read, and I suspect that the reader will too.

Question One

> Would you say that your principle discoveries have been the result of deliberate endeavour in a definite direction, or have they arisen, so to speak, spontaneously? Have you a specific anecdote of a moment of insight/inspiration/illumination that would demonstrate this? [Hadamard # 9]

[17] Permission to do this was sought and subsequently approved by the Office of Research Ethics. See appendix A and B.

The responses to this question, for the most part, were in one of three forms. The first of these was centred on discussion of the role of deliberate work in a specific direction.

> *I have found that exacting work is necessary to become thoroughly acquainted with a problem before there is much hope of getting a solution.*
>
> - Connor[18]

> *My main way of proceeding is trying to get an understanding, rather than trying to solve a problem. I first of all want to understand how things stand together, and what symmetries they have – while keeping in the back of my mind questions I care about, so as to be alert if in the whole panorama a way of attack presents itself.*
>
> - Pierre Rene Deligne

Both of these anecdotes speak of thoroughly getting to know the problem before a solution can even be contemplated. More importantly, however, is that both Connor and Deligne speak of it in a prescriptive fashion; this is not just something that *happens* in their problem solving efforts, but something that they *do* to help it along. This resonates with Mason et al.'s (1982) emphasis on *specializing*, discussed in chapter two, as an initial tool for problem solving. It also speaks very clearly towards the notion of initiation that Hadamard (1945) referred to as the first step in the process of invention. Again, however, this is more than aimless casting about in search of a solution that can so often happen in problem solving situations. Connor, in particular, refers to it as "*exacting work*", while Deligne speaks of "*want[ing] to understand how things stand together, and what symmetries they have*".

[18] Connor is an applied mathematician who preferred to remain anonymous.

A notion similar to *deliberate effort* is that of *persistent work*. This question prompted a number of comments from the participants with regards to this notion. However, such comments also arose within the responses to other questions, and were so prevalent, that I have devoted an entire section of the discussion to it near the end of the chapter.

A second form of response to question one had to do with the formation of ideas occurring as a *change in direction*.

> *My principal discoveries have arisen "spontaneously." Also, nearly every one was perceived as a change of direction, in fact their topics were first reputed to wander all around. But every so often there was a spurt of after-the-fact "self-organization" and reorganization that affected future progress.*
>
> \- Benoit B. Mandelbrot

> *My discoveries have all arisen indirectly, not by direction. They have come from my asking myself questions about something that I think is mysterious or incompletely understood, and trying to get to the bottom of it.*
>
> \- Michael Atiyah

Mandelbrot speaks of this change in direction directly and as if it occurs somewhat incidentally. Atiyah, on the other hand, speaks of it more tangentially and, I believe, at a level of how he comes to decide on a topic rather than how he makes a discovery. Nonetheless, both mathematicians clearly speak of 'doing' mathematics as something that is not vectored and unidirectional. This is consistent with my discussion of the dichotomous role of mathematics that I presented in chapter two.

The final type of responses has to do with the indirect role of deliberate effort in the solving of mathematical problems.

> *Serendipity is very important, but it only works if the ground is suitably prepared. I guess the reason is that the "straightforward" discoveries are easy to find, even if they may need a little sweat, and thus have been found by people working in the field before you.*
>
> - Peter J. Huber

> *Thus while you can turn the problem over in your mind in all ways you can think of, try to use all the methods you can recall or discover to attack it, there is really no standard approach that will solve it for you. At some stage, if you are lucky, the right combination occurs to you, and you are able to check it and use it to put an argument together. Thus in answer to your first question, deliberate endeavour is required, but it is rarely sufficient for important discoveries. In fact, any problem in which you can see how to attack it by deliberate effort, is a routine problem, and cannot be an important discovery. You must try and fail by deliberate efforts, and then rely on a sudden inspiration or intuition or if you prefer to call it luck.*
>
> - Dan J. Kleitman

Huber's comment in particular, resonates with the idea that there is a preparatory stage to the discovery process in which the "*ground is suitably prepared*". This stage, although, not producing valid solutions is still "*required, but is rarely sufficient*" as Kleitman points out.

The very notion as to what constitutes a problem was also called into question. The definition that was presented in chapter two, although circular in nature, encapsulates the essence of a problem. A problem is something that is 'problematic' (Perkins, 2000). This idea is consistent with the comments of both Huber and Kleitman. Huber talks about "*straightforward discoveries*" and how they are easy. In doing so he implies what Kleitman more explicitly states, "*any problem in which you can see how to attack it by deliberate effort, is a routine problem, and cannot be an important discovery*".

The most striking element in these comments, however, is the mention of 'luck' and 'serendipity'. Although I was familiar with the *chance hypothesis*, presented in Hadamard's (1945) work, I had not given much credence to it prior to receiving the responses from the mathematicians. Recall that at the outset of my research into the AHA! experience I was determined to learn how to manufacture AHA!'s in a controlled and clinical manner. With such a goal in mind I failed to appreciate the role that chance plays in the phenomenon. However, as will be seen in the responses to question two, many mathematicians greatly value the contribution that chance makes to the process of invention.

Before I move onto the next question, however, I will address the second part of question one. I asked the participants to provide a specific anecdote that demonstrated the role of insight, illumination, or inspiration. For reasons not entirely clear to me, many of the respondents chose to ignore this part of the question. Of those that did respond to it many provided only partial anecdotes as to what it was they were doing at the moment of discovery: driving, sleeping, cooking, or showering. However, there was one particularly interesting account of such an occurrence offered by Dusa McDuff.

> *In my principle discoveries I have always been thinking hard trying to understand some particular problem. Often it is just a hard slog, I go round arguments time and again seeking for a hole in my reasoning, or for some way to formulate the problem/structures I see. Gradually some insights builds and I get to "know" how things function. But the main steps come in flashes of insight; something clicks into place and I see something clearly, not necessarily what I was expecting or looking for. This can occur while I am officially working. But it can also occur while I am doing something else, having a shower, doing the cooking. I remember that the first time I felt creative in math was when I was a student (undergrad) trying to*

find an example to illustrate some type of behaviour. I'd worked on it all the previous evening with no luck. The answer came in a flash, unexpectedly, while I was showering the next morning. I saw a picture of the solution, right there, waiting to be described.

- Dusa McDuff

Although the details of the mathematics are missing the essence of the experience remains. This is an important point that I will come back to in the closing remarks of this chapter, and again in chapter eight.

Question Two

How much of mathematical creation do you attribute to chance, insight, inspiration, or illumination? Have you come to rely on this in any way? [Hadamard# 7]

As mentioned earlier, I hesitated to include this question in the survey because I felt it spoke too directly at the idea of illumination, an aspect of the AHA! experience that I wished would emerge from the data. I chose to leave it in, however, because I felt that it provided the participants with a rich set of descriptors to be used in their discussion. As anticipated, they made good use of these terms in their bid to articulate the rapid and unanticipated emergence of a new idea or solution. Also, as anticipated, they do so with little consistency of usage between mathematicians. Of the three – *insight, inspiration,* and *illumination* – inspiration was the most commonly used for this purpose.

An example is one of my most famous discoveries, namely the symplectic reduction theorem. I was writing my textbook on mechanics and basically just assembling work of others on the geometric view of mechanics. It may have been in the shower that it just occurred to me that the work of some of the classical authors could be generalized in a certain way. As my anecdote indicates, it is not just chance, but rather inspiration in the presence of lots of

> *surrounding information. The surrounding information is really crucial, I believe. For me, this has worked in situation after situation. Sometimes, the old "Eureka" phrase is a good description. For example, I can be talking to a colleague or my wife or eating breakfast and suddenly, like a voice from the blue, I get told what to do. Hard to explain.*
>
> \- Jerry Marsden

> *Given this obsessive preparation, there are several kinds of worthwhile contributions one can make. One type is the kind where a bridge gets built between two apparently unrelated fields. Another is the kind where an expected result is established in the core of an existing area by very hard work, involving the creation of new levels of structure in an unexpected way and then exploiting that structure. Another is the opening up of a new subject for mathematical investigation. I would say that the first and third kinds of contribution are the most likely to feel to the researchers as a "flash" of inspiration. The second is more likely to feel as a result of enormous labour, with a lot of different avenues explored and tested.*
>
> \- David L. Donoho

In his explanation, Marsden also contributes a new idea to the discussion of the problem solving process, that being '*surrounding information*'. This will become an important part of the discussion on *chance* that is yet to come. Meanwhile, Donoho gives a characterization of discovery in different areas of the *mathematical landscape*. The idea of mathematical landscapes is an important, and reoccurring, one. However, I defer discussion of this until later on in this chapter where it is treated as its own theme.

Enrico Bombieri gives a nice account of a moment of illumination. In doing so he shows how he sees this as being a rapid usage of the intuition. This understanding of the intuition as operating like illumination is consistent with Fischbein's classification of intuition according to situational context within which it is used (see chapter one).

> *That was the initial intuition, and in five minutes I knew it could be done and all the consequences it would entail. In conclusion, I think that for my best work I need intuition (or illumination, if it comes really suddenly) and also determination in reaching a goal [..] there have been occasions in which ideas came to me almost by chance or almost by themselves. For example, reading a paper one may see almost in a flash how to remove a stumbling block.*
>
> - Enrico Bombieri

Faltings stipulates that, by definition, insights appear in a flash. This is in contradiction to Dusa McDuff's usage of insight by stating that they build gradually.

> *In some sense all insights come suddenly, usually in some impure form which is clarified later.*
>
> - Gerd Faltings

George Papanicolaou manages to speak of the phenomenon without using any of the words *insights, inspiration,* or *illumination*. Meanwhile, Solomon Feferman uses all three, not making it clear if he sees them as interchangeable or indistinct.

> *Sometimes after thinking about a problem a complete solution comes out as if it had been worked out in detail before. I am not sure how this happens, perhaps because some methods and tools do become second nature to us after a while.*
>
> - George Papanicolaou

> *Usually, specific endeavour in a definite direction, but often the difficulties met in the process are overcome only through insight/inspiration/illumination. But some lines of pursuit came about through the latter, with the sudden idea that it might be possible to do something of a certain kind and/or in a certain way. Hard to say how much. Significant amount. Of course I rely on it; the more it works the more you rely on it.*
>
> - Solomon Feferman

The most consistent usage was with respect to *chance*, an aspect of problem solving that I have come to see is highly valued. There are two main usages of *chance*, or *luck* as it is often referred to. The first of these is what I have come to call *intrinsic chance*. *Intrinsic chance* refers to a belief that you are lucky that an idea came to you. This is nicely articulated in Kleitman's comments.

> *And relevant ideas do pop up in your mind when you are taking a shower, and can pop up as well even when you are sleeping, (many of these ideas turn out not to work very well) or even when you are driving. Thus while you can turn the problem over in your mind in all ways you can think of, try to use all the methods you can recall or discover to attack it, there is really no standard approach that will solve it for you. At some stage, if you are lucky, the right combination occurs to you, and you are able to check it and use it to put an argument together.*
>
> - Dan J. Kleitman

The idea that "*the right combination occurs to you*" is reminiscent of Poincaré's description of ideas mobilizing, and crashing together to form new combinations (see chapter one), although he did not directly refer to luck or chance in his discussion.

The second type of chance that emerged in the responses of the mathematicians is what I call *extrinsic chance*. *Extrinsic chance* refers to a situation in which external situations conspire to provide the solver with a fortuitous piece of information or stimulus. This is the more common usage of *luck* and *chance* and was seen in a number of responses including those of Mandelbrot, Askey, and Connor.

> *Do I experience feelings of illumination? Rarely, except in connection with chance, whose offerings I treasure. In my wandering life between concrete fields and problems, chance is continually important in two ways. A chance reading or encounter has often brought an awareness of existing mathematical tools that were new to me and allowed me to return to old problems I was previously obliged to leave aside. In other cases, a chance*

encounter suggested that old tools could have new uses that helped them expand.

- Benoit B. Mandelbrot

Having the right background to appreciate a special result or problem and see how a more general one can be built from this is sometimes a matter of chance. I would say that what was most important for me was problems which came up in trying to solve other problems and these led me to a few different areas.

- Dick Askey

But chance is a major aspect: what papers one happens to have read, what discussions one happens to have struck up, what ideas one's students are struck by (never mind the very basic chance process of insemination that produced this particular mathematician).

- Connor

However, chance encounters are but passing opportunities. It is up to the individual mathematicians as to what happens then. First, and foremost, they must recognize the opportunity that has been presented to them as attested to by Wendell Fleming.

Chance will favour only those who are prepared.

- Wendell Fleming

In this statement he is echoing the sentiments of Louis Pasteur; "*Chance favours the prepared mind*" (Koestler, 1964). However, recognition is not enough. The opportunity must be seized if it is to be realized.

Chance plays a role, but the key thing is to grab the chance. Here insight or intuition are very important.

- Michael Atiyah

Question Three

Could you comment on the differences in the manner in which you work when you are trying to assimilate the results of others *(learning mathematics) as compared to when you are indulging in personal research (creating mathematics)? [Hadamard # 4, 15]*

There were two themes that emerged in the responses to this question. The first has to do with the role that attention to details plays in the learning of mathematics.

> *First, read widely, superficially and indiscriminately, so that you know where to look for something if the need arises. Usually, I pushed from the special to the general.*
>
> - Peter J. Huber

> *Of course I like to follow things generally and this is important in providing a wide base from which to explore.*
>
> - Michael Atiyah

> *However I usually try whether I can find my own proofs for the assertions before I dig into their details.*
>
> - Gerd Faltings

> *There is not much difference. More precisely, I seldom study or learn mathematics in detail.*
>
> - Benoit B. Mandelbrot

> *Understanding others is often a painful process until one suddenly goes beyond the details and sees whole what's going on. Many things are so difficult or so foreign that that never happens. Teaching other people's mathematics is the best way to achieve understanding.*
>
> - Solomon Feferman

The overwhelming sentiment on this issue was that the details were to be ignored, at least initially, in the learning of mathematics. Understanding others' work is often difficult (as expressed by Feferman above). It is not until you push beyond the details and deal with the mathematics on a larger and more general scale that you can begin to make sense of it. This idea runs completely counter to the philosophy that governs both the teaching of mathematics as well as the creation of its curricula, and has powerful implications for what it means to teach mathematics. In contrast, creating your own mathematics is a relatively easy task, as Bombieri so eloquently articulates.

> *Reading a paper by another mathematician for me is comparable to a difficult hike in the mountains with the help of a guide, while in creating mathematics you are in a more familiar territory and a guide is not needed, you can follow your own path based on your experience and feeling.*
>
> \- Enrico Bombieri

Again, this runs counter to perceptions of mathematics and how it is taught.

The second, and equally implicative result that emerged from the responses to this question has to do with the role that *talking with others* plays in the communication of mathematical knowledge.

> *Most of my work – including the best – has been stimulated by ideas from my colleagues. Once I hear something interesting, I get going.*
>
> \- Brad Efron

> *I assimilate the work of others best through personal contact and being able to question them directly. I have come to rely on this more and more as the literature is so huge that trying to read it all is almost impossible. In this question and answer mode, I often get good ideas too. In this sense, the two modes are almost indistinguishable.*
>
> \- Jerry Marsden

> *I get most of my real mathematical input live, from (good) lectures or one-on-one discussions. I think most mathematicians do. I look at papers only after I have had some overall idea of a problem and then I do not look at details.*
>
> \- George Papanicolaou

Like the comments on details, the idea that one learns most effectively through talking has deep implications for what it means to teach mathematics.

Question Four

> *Have your methods of learning and creating mathematics changed since you were a student? How so? [Hadamard # 16]*

The responses to question four complement those of question three in that they continue on the theme of avoiding details and learning through talking.

> *My way of reading mathematics has not changed much. I read very quickly, trying first to understand the key points, and only afterwards fill in details if needed.*
>
> - Enrico Bombieri

> *As a student one has time to study in depth and read textbooks. Later this is hardly possible. But one compensates by learning from talking to one's colleagues.*
>
> - Michael Atiyah

However, this also manifested itself in some explicit comments against the usefulness of reading mathematics.

> *I have also learnt that it is important for young mathematicians to work on his/her own problems as early as possible ('mathematics is a young man's game'). Knowledge is needed but not enough, so that book learning should not be emphasized too much.*
>
> - Ulf Grenander

> *As a student I relied a lot more on books and papers (the classics). Now I hardly read books in the fields I know, and often I do not like the books I read (for example in financial math or imaging and random media) because very few of them really contribute to the subject.*
>
> - George Papanicolaou

Ulf Grenander's comments, in particular, are very telling of a philosophy about learning mathematics in which the work of others can interfere with one's own coming to know the subject matter. This philosophy was highly touted by the mathematician R. L. Moore, after whom a method of mathematics instruction was named; the *Moore Method* (Jones, 1977). Moore believed that the only mathematics worth knowing was that which you had yourself created. As such, he forced his students to sign contracts agreeing that they would not seek help from sources outside of their own cadre of peers, including other mathematicians, journals, and textbooks.

Question Five

> *Among your greatest works have you ever attempted to discern the origin of the ideas that lead you to your discoveries? Could you comment on the creative processes that led you to your discoveries? [Hadamard # 6]*

Recall that this question was chosen to be included in the survey because I was hoping to solicit from the participants any ritualistic beliefs and practices that they engaged in for the purposes of enhancing their creative processes. In this regard, I was not disappointed. Some of the participants speak to a process of finding ways to allow the mind to become distracted, presumably to allow the problem to incubate.

> *Things are more complicated. If I was stumped by a problem and seemed to walk around a solid smooth, blank wall, then I would consciously stuff the problem into my subconscious, do something entirely different, and hope for some revelation to surface in due time (it often did).*
>
> \- Peter J. Huber

Others focused on strategies for avoiding the details of the situation that, as discussed above, can be distracting to the process of really getting to understand the mathematics.

> *I learned from two colleagues of mine (rather late in my mathematical life) that it is a waste of time to spend endless consecutive hours working on a problem, or filling page after page*

with calculations and scribbles. Get the basics of the problem firmly and thoroughly into the head. After that, an hour or two each day of thinking on it is all that's needed for progress. For sure, what patterns are formed in one's head is important and, perhaps, one can influence that. Certainly, thinking vaguely about one's problem during a math talk can have a beneficial effect. For that reason, I started some 20 years ago to ask students (and colleagues) wanting to tell me some piece of mathematics to tell me directly, perhaps with some gestures, but certainly without the aid of a blackboard. While that can be challenging, it will, if successful, put the problem more firmly and cleanly into the head, hence increases the chances for understanding. I am also now more aware of the fact that explaining problem and progress to someone else is beneficial; I am guessing that it forces one to have the problem more clearly and cleanly in one's head.

- Connor

I would like to mention that I find it almost impossible to have a creative thought while sitting at a desk. To the extent I "discover" things it is almost always while walking or pacing. Of course I can "work things out" at a desk, or on the computer, but to really "turn things over in my mind" I have to walk around.

- Stephen[19]

Finally, Lars Gaarding offered as a means for enhancing the creative process the construction of what he calls a *net*.

In the first place the creator must have what I call a net. This is a connected collection of facts, results, guesses and so on that the creator keeps in his head and to which he has immediate access. To build a net requires time, work, and interest in and love of the material. A creator must live with his net and more or less think of it all the time. The state of the net depends on the brain capacity and quickness of thought of the creator. Thinking about the net can be precise, dreamlike or haphazard or systematic or in pictures or not in pictures. All the ways of the human brain may be useful. A net is necessary for a creator but not sufficient. A net offers its owner the possibility to create something new and thus become a true creator. The above applies to engineers, philosophers, physicists, chemists, writers, artists and so on. Creativity in mathematics is considered to be mysterious by most people because they cannot imagine what a mathematical net could contain. Most of them even shy away from the opportunity to create a small net from the

[19] Stephen is a Field's Medallist who preferred to remain anonymous.

> *mathematics taught in the schools. What I have written here is true but unfortunately not scientific.*
>
> - Lars Gaarding

For all intents and purposes, this net seems to be synonymous with the idea of a *repertoire* as put forth by Schön (1987) and discussed in chapter two with respect to problem solving by design. However, as Gaarding points out, this net is necessary but not sufficient. Unfortunately, he does not expand on this point to include exactly what he sees as being necessary in his discussion.

Question Six

> *How do you know that you have had an AHA! experience? That is, what qualities and elements about the experience set it apart from other experiences.*

In many ways the responses to this question were, in one form or another, definitions of the AHA! experience, all of which echo parts of the definitions discussed in chapter one. In some cases the participants focus on the *suddenness* with which an answer appears as the defining characteristic.

> *When, after some considerable, quite non-productive effort, usually while not at all consciously working on the problem, there appears, for no apparent reason, in your brain the answer to that problem – that's the AHA! experience. It can also happen when you are working on the problem, and then it is the apparent suddenness that generates the AHA! experience.*
>
> - Connor

Others discuss the feeling of *certainty* that accompanies the AHA! experience.

> *When an idea comes up that solves a hard problem that has been with you for a while you just know it is IT. You may need to do a lot of work to check that things do work out as you expect and this takes time. In some cases the real results are not quite what you*

wanted but it was still a good idea you had. In a few cases the results do work out all the way.

- George Papanicolaou

Still, others focus on the *significance* of the discovery as the key element in the experience.

What I think you mean by an AHA! experience comes at the moment when something mathematically significant falls into place. This is a moment of excitement and joy, but also apprehension until the new idea is checked out to verify that all the necessary details of the argument are indeed correct.

- Wendell Fleming

However, most use a combination of *suddenness, certainty,* and *significance* in their descriptions of the phenomenon and what sets it apart.

It is, in my experience, just like other AHA! experiences where you suddenly "see the light". It is perhaps a little more profound in that you see that this is "important". I find that as one gets older, you learn to recognize these events more easily. When younger, you often don't realize the significance of such an event at the time.

- Jerry Marsden

A Eureka experience (I prefer this term) is characterized by suddenly realizing that you have found the missing piece of the jig-saw puzzle. Once found it is obviously right.

- Michael Atiyah

This is very subjective. I would say that crank science is often fueled by people who had some kind of "sudden vision" which for them becomes "absolute truth". Anyway, I can compare the AHA! experience to putting together a very complicated puzzle without a blueprint, and suddenly you realize what it should be, and the pieces fall in the proper slot instantly. One does not need to put all the pieces in their proper places. Once you get the idea, the vision where exactly the bridge should be built, you know right away the litmus test to apply in order to confirm it.

- Enricho Bombieri

One participant, Peter J. Huber, offers a deeper and more analytic response to the question.

Your questions on AHA!/EUREKA! experiences appear to refer to the sudden, seemingly miraculous appearances of a non-trivial insight. To my regret I do not remember having had any such experience in mathematics, whose intensity and instantaneity would make me jump out of the bathtub! I guess the closest I did come to what you might classify under these two words would occur in two separate steps. Perhaps an oil explorations simile is more appropriate. First I would have a promising, brilliant(?) idea (the AHA! event) which would induce me to drill. But the eureka event ("I found it!") at best would come hours or days later, if and when the oil would begin to gush forth. That the idea had been brilliant and not merely foolish would be clear only in retrospect, after attempts to verify and confirm it. And later on one tends to suppress and forget foolish ideas because they are embarrassing (but they are indispensable companions to the brilliant ones!).

- Peter J. Huber

In doing so, he not only presents a distinction between the AHA! experience and EUREKA! experience which I had not considered, but he also gives insight into the occurrence of '*foolish ideas*'. Huber clearly views the AHA! as the experience that accompanies the initial, raw, and unverified idea. Meanwhile, he sees the EUREKA! as the experience that accompanies the realization, or verification, that the initial idea has borne fruit. This is in line with the definition of the AHA! experience, presented in chapter one, as the process of invention minus the consideration of the product and, as such, an experience which is not lessened by inaccuracy or incompleteness.

Finally, Henry McKean offered a very succinct statement that speaks to how he sees the understandings gained from AHA! experiences as being no different than understandings gained from other thinking processes.

No, I don't find it different from understanding other things in life except that most other things of any real importance have to be re-understood over and over, for years!

- Henry McKean

Question Seven

> *What qualities and elements of the AHA! experience serve to regulate the intensity of the experience? This is assuming that you have had more than one such experience and they have been of different intensities.*

Unlike question six, above, this question spawned a much wider array of responses. It seems that what regulates the intensity of the AHA! experience is much less definite than what separates it from other mathematical experiences. A theme that did emerge, however, was the role that *time* played in the intensity.

> *The harder and more prolonged the prior work, and/or the more sudden and unexpected the insight, the more intense is the AHA! experience. To be sure, it is also possible (as I know from ultimately sad experience) to have AHA! experiences based on what turn out to be false insights. Certainly it is standard among mathematicians to enjoy these AHA! moments while they last and postpone for a bit (e.g., until the next day) the necessary checking of the insight.*
>
> \- Connor

> *It depends how long one has worked, how many silly mistakes one made. Here's a good quote for you from Gelfand. Povzner says: ``Gelfand cannot solve difficult problems. He only solves simple problems''. Do you see?*
>
> \- Henry McKean

In his comments, Connor not only reflects on the role of time, but also revisits the role of suddenness, this time with respect to the intensity of the AHA! experience. He also contributes the idea that intensity of the conscious effort contributes to the eventual intensity of the AHA! experience. McKean also mentions the length of time that has been spent, as well as adding the comment about the number of silly mistakes. His quote with regards to Gelfand I find insightful and is reminiscent of a Gestalt philosophy of problem solving, discussed in chapter two, in which the problem is turned and turned until a 'simple' way of solving it is seen. This also introduces the dimension of *simplicity* as one of the regulators of intensity as described by George Papanicolaou.

> *In the three or four cases where a clear advance was made (as pointed out last year in item 5) the degree to which the idea worked out as hoped for is a measure of its importance and the satisfaction that it gives. Sometimes it is the simplicity of the idea or the ultimate simplicity of the results it gives. I have also noted that often the really important idea in solving a problem may be visible only to a very small number of readers. It is hard to see where the pressure points were in a well-solved problem, not because they are hidden or not spelled out but because of the relative simplicity and effectiveness of the approach followed.*
>
> - George Papanicolaou

In this passage, Papanicolaou mentions not only the simplicity of the initial idea, but also the simplicity of the results it produces. This can often result in a simplification of the problem, or even the field within which the problem is embedded. The result of such simplification is that the initial hurdles that blocked progress may become indistinguishable to those who view the mathematics in its completed form. Simplification is not the only way in which an AHA! experience can affect understanding.

> *Basically the intensity of the experience depends on what kind of understanding it brings. For example, you may be working on a certain problem and suddenly you see how to do it, and realize that the original problem was only a small part of what one may call "the right question to ask" and solving the original problem brings immediately the solution of the new question, which is of course a most gratifying experience. In other cases, you realize that you have opened the door to a new, totally unexplored direction which you know will produce a great harvest of results. These are probably the most intense experiences. This is often associated with a dramatic simplification of the way in which such problems were originally studied. For a non-mathematical example, think of the double helix. In any case, the experience of an AHA! moment is*

> *always associated with a clarification and simplification of understanding, even if at a later stage one may encounter major technical problems in order to carry the work to the very end.*
>
> - Enrico Bombieri

Bombieri adds to his discussion on simplification and clarification the possibility that an AHA! experience may also create a broader understanding by opening the door to previously unknown areas of mathematics. This is very similar to Donoho's comments about "*opening up of a new subject for mathematical investigation*" in his response to question two.

Another theme that reappeared in the responses to this question was with regards to *significance*.

> *The intensity is of course regulated by the magnitude of the new insight, and/or the desperateness before. Also I should mention that these experiences are not so uncommon, but many of them do not last long because often the new insight later turns out to be false.*
>
> - Gerd Faltings

> *The depth of the experience depends on how profound the ultimate result is. Sometimes the experience is easy to remember in retrospect because it happened during a particular walk, either alone or in company.*
>
> - Michael Atiyah

Faltings speaks of the "*magnitude*" of the new insight while Atiyah is talking about the significance of the "*ultimate result*". Atiya's comment with regard to how "*profound*" the "*ultimate result*" is, as well as Fleming's response to question six in which he speaks of a new find as being "*mathematically significant*" are both statement of absoluteness. That is, they refer to the significance of the final outcome, something that cannot be ascertained until after a verification process. This is somewhat contradictory and deserves closer attention. I mention this here only to highlight it. The closer attention it deserves will be provided in a subsequent section in which the role of significance is discussed as an emerging theme.

This concludes my discussion of the responses with respect to the seven questions that were sent out. As mentioned earlier, however, there were themes that emerged from the data that transcended the individual questions and appeared over and over again throughout the responses. Some of these themes have already been alluded to in the analysis of the data thus far. In what follows I present a deeper discussion of these themes, as well as of themes not yet encountered.

Emergent Themes

In reading anecdotal data, themes will often emerge that transcend the initial organization of analysis that has been chosen by the researcher. Sometimes these themes are obvious and self-explanatory within the context of the data itself. Other times, however, the emerging themes are more subtle, and require a level of analysis and explanation to bring them into sharper focus. The themes that emerged from the data in this study are no different. There are themes such as 'sleep' and 'persistent work' that are obvious and there are themes such as 'significance' that are more subtle. As such, my presentation of the various themes is accompanied by varying degrees of discussion and analysis as necessary to bring them into sharper focus. In doing so I, again, rely heavily on anecdotal comments. Because of the extensiveness of this chapter I have also chosen to repeat occasionally a comment from earlier in the chapter rather than referring to it from such a great distance.

Sleep

The first theme has to do with sleep and the role it plays in doing mathematics and AHA! experiences. Many of the participants commented at one time or another about the phenomenon of waking up to a solution, either in part or even fully formed. I was aware of this phenomenon from Hadamard's work (Hadamard, 1945), who found it sufficiently interesting to include it as part of his survey:

> *Have you ever worked in your sleep or have you found in dreams the answers to problems? Or, when you waken in the morning, do solutions which you had vainly sought the night before, or even days before, or quite unexpected discoveries, present themselves ready-made to your mind?* (Hadamard, 1945, p. 138)

As such, I was not altogether surprised to see comments of this nature cropping up in the responses to my survey.

> *While at a meeting in Philadelphia, I woke up one morning with the right idea.*
>
> - Dick Askey

> *I'm convinced that I do my best work while asleep. The evidence for this is that I often wake up with the solution to a problem, or at least with a clear idea of how to proceed to solve it.*
>
> - Charles Peskin

> *One of the most intense such experience I had actually turned out to be nonsense. It occurred in a dream in which I really thought that I got insight into a really hard problem. When I got up, I rushed to my desk to think and after an hour realized that it was all gibberish. But it was quite intense. I find, in general, ideas that come to you "in*

the shower" are more reliable than those that come to me in my sleep (which does not happen very often).

- Jerry Marsden

While Askey and Peskin discuss the fruitfulness of the process of solving mathematical problems in their sleep Marsden comments that awakening with an answer, although very tempting, is often quite misleading.

Christodoulou presents a slightly different role of sleep.

I do not attribute much to chance; I would attribute all to insight and illumination. In regard to illumination, I would like to add that in my case the best instances have been at night when I am lying in bed, somewhere between consciousness and sleep. It is during these times that I have the greatest power of concentration when all else except my subject lose reality.

- Demetrios Christodoulou

The suspension of conscious thought in that state between being asleep and being awake is, for him, a fertile venue for new ideas to form.

In general, discussions of AHA! experiences and sleep are reflective of Hadamard's discussion of unconscious work. Sleep, and the time just before sleep, are times during the day where the unconscious mind is unhampered by the distractions of conscious thought. It seems logical, therefore, that such times would produce more such AHA! experiences. However, Marsden's warning should be heeded, unconscious (or sleep) work does not mean good work.

Metacognition

The next theme I present has been given the title 'metacognition'. The comments under this heading all share the characteristic that they comment in some fashion on the participants' reflective analysis of their thinking process. In some cases this manifests itself as thinking strategies.

> *I try to build/find structure and cohesion in which I am looking at. I think math is a language; one sees things with some internal eye and needs to find a language to express this.*
>
> - Dusa McDuff

> *Rather, I stop and ask myself: What did I really find? What is next? Sometimes this is the first step for real progress.*
>
> - Enrico Bombieri

> *A shorter analog: after a partial step, wondering: what does this argument really mean.*
>
> - Pierre Rene Deligne

Each of these mathematicians – McDuff, Bombieri, and Deligne – uses their respective strategies in solving problems. McDuff looks for structures and cohesion of these structures. This is indicative of visual thinking and is consistent with her later comment that her "*imagination is quite visual (though not as much as some others I know of). Other mathematicians have a feel for algebraic structure, actual equations turn them on – not me...*" McDuff thinks with pictures. Bombieri and Deligne's strategies are very different from that of McDuff, but very similar to each other. They use their strategy to stop their progress and question what they have found, or what they are constructing. Schoenfeld (1985) refers to this as 'control'. What they all have in common, however, is that they are all aware of how they think, and they use it.

Other mathematicians' reflections are more of a commentary of their thinking process than a description of a thinking tool.

That is, unless you are aware of a problem and have given it some thought, it is very unlikely that you will solve it: and if you did, how would you know you did, if you were really unaware of the problem.

- Dan J. Kleitman

My passion for the history of ideas is boundless and I go to endless lengths, not to hide the influences from which I benefited, but to understand and express them thoroughly. To me, the value of a thought combines its novelty and difficulty with the depth of its roots. The greatest thrill is to add to streams of ideas that already have a long and recognizable past.

- Benoit B. Mandelbrot

I have become more suspicious than I used to be about the originality of my ideas.

- Connor

Kleitman philosophises about the role of conscious work in the problem solving while both Connor and Mandelbrot reflect on the origin of their ideas.

Mathematical Landscape

The participants often turned to very rich descriptors and metaphors in their comments. In several cases these descriptions refer to the "*mathematical landscape*." I have already alluded to this in my discussion of Donoho's response to question two in which he discusses how "*a bridge gets built between two apparently unrelated fields*". Bombieri is much more explicit in his comments.

My approach to research consists in looking to the mathematical landscape, taking notice of the things I like and judge interesting and of those I don't care about, and then trying to imagine what should be next. If you see a bridge across a river, you try to imagine what lies on the other shore. If you see a mountain pass between two high mountains, you try to imagine what is in the valley you don't see yet but secretly know must be there.

- Enrico Bombieri

He not only uses the metaphor of building a bridge as Donoho does, but he also imagines what lies on the other side. For Bombieri this is more than just a metaphor, it is also a thinking tool that he uses in research. He continues with the metaphor in later comments where he explains his Platonic view of mathematics.

> *My attitude towards mathematics is that most of it is lying out there, sometimes in hidden places, like gems encased in a rock. You don't see them on the surface, but you sense that they must be there and you try to imagine where they are hidden. Suddenly, they gleam brightly in your face and you don't know how you stumbled upon them. Maybe they always were in plain view, and we all are blind from time to time.*
>
> \- Enrico Bombieri

Connor also mentions the landscape, but he does so from a descriptive perspective, where he views it rather than traverse it like Bombieri did.

> *However, all during that process, `insights' do appear seemingly spontaneously. However, this seems to me to be akin to artists looking at a landscape and being amazed by how interesting this or that view is – an amazement that ignores the fact that their artistic pattern recognition will only make them aware of certain views, namely those that are striking.*
>
> \- Connor

Burton (1999a), in her work with mathematicians, also found that the mathematical landscape was a prevalent metaphor, both descriptively and prescriptively.

Although the mathematical landscape was a prevalent metaphor in the mathematicians' responses, it wasn't the only one. Art provided inspiration for the formation of metaphors for both Brad Efron and Enrico Bombieri.

> *At first I'm terribly confused, but after awhile I chip away at my wrong ideas until I'm left with an answer. So I think I'm working in the sculptor mode, rather than the inspired painter.*
>
> \- Brad Efron

I would say that my attitude towards mathematics is more that of a problem solver than of a builder of theories. I can paraphrase this by saying that I am not an architect or urban planner, rather more of a painter working small paintings depicting what the inspiration leads him.

- Enrico Bombieri

Gaps

Along with a mathematical landscape comes *gaps* in knowledge; places in the terrain that have not yet been explored.

Certain gaps in knowledge needed to be filled and my main role was to feel that these gaps could be filled.

- Dick Askey

However, this is not the only context where mention of *gaps* appeared in the mathematicians' responses. There were also several comments pertaining to gaps or holes in thinking, reasoning, and arguments.

In my discoveries the general direction or topic has been planned, of course. However after that one has to search the terrain until one finds an opening (or gives up), and where that is cannot be planned.

- Gerd Faltings

I go round arguments time and again seeking for a hole in my reasoning, or for some way to formulate the problem/structures I see.

- Dusa McDuff

You look in your mind for something and come back with something else which may, with enough looking and some luck, be just what you need to fill the gap in your argument.

- Dan J. Kleitman

The existence of these gaps is indicative of a related theme that emerged, the theme of failure and wrong ideas.

Failures and Wrong Ideas

Recall that Hadamard had been quite critical of the original survey published by Édouard Claparède and Théodore Flournoy because they failed to ask questions pertaining to failures directly. Recall also that he felt that only "first-rate men would dare to speak of" (p.10) such things. With this in mind, I chose to survey "first rate" mathematicians. I also chose, as Hadamard did, to not ask any questions directly related to failures. Nonetheless, many of the mathematicians chose to comment on their wrong ideas and false leads.

> *Also I should mention that these experiences are not so uncommon, but many of them do not last long because often the new insight later turns out to be false.*
>
> \- Gerd Faltings

> *My usual mode is to jump in and compute (I cannot really think without a pen in hand). Then having computed fast and probably wrong, I find that this particular calculation would not have done what I wanted anyhow, so I throw it out and start over. Sometimes I simply repeat what are at bottom the same stupidities for weeks, and though this looks useless on the face of it, I get familiar with the question and learn a few tricks. Of course I know already what I want to come out, mostly by analogy with old things of my own or others, and I'm looking for the mathematical mechanism that makes it work … [the intensity of the AHA! experience] depends how long one has worked, how many silly mistakes one made.*
>
> \- Henry McKean

> *And relevant ideas do pop up in your mind when you are taking a shower, and can pop up as well even when you are sleeping, (many of these ideas turn out not to work very well) or even when you are driving. You must try and fail by deliberate efforts, and then rely on a sudden inspiration or intuition or if you prefer to call it luck.*
>
> \- Dan J. Kleitman

> *But I should say that my subconscious usually would present a novel way of attack; if it presented a ready-made "solution", it often was quite wrong.*
>
> \- Peter J. Huber

> *Also, one must expect to consider many ideas which turn out later to be failures.*
>
> \- Wendell Fleming

> *At first I'm terribly confused, but after awhile I chip away at my wrong ideas until I'm left with an answer.*
>
> \- Brad Efron

A qualitatively different type of mistake is the one pointed out by Connor in which he discusses the false assumption that one's ideas are original.

> *I imagine that a mathematician's brain is similarly engaged in a ceaseless search for striking patterns in the ever-changing stream of ideas, making the mathematician aware of only the patterns found striking by the mathematician's pattern recognition process. I am certainly aware of the very moment of such `insight' on more than one occasion. In fact, this moment can be so striking that it has led people (me included, I am sorry to say) to believe they are the first to have a certain idea even when it can be established (afterwards) that they heard or read that idea earlier; the moment of `seeing' it, i.e., of recognizing the pattern, is much more powerful than the moment of hearing or reading some fact to which one has, at that point, no immediate connection.*
>
> \- Connor

I found this to be an interesting and important statement for two quite different reasons. The first is that it highlights the impact that the sudden appearance of an idea can have on otherwise rational thought. It is perhaps for just such a reason that Deligne and Bombieri have developed a thinking strategy that gets them to question exactly what it is they have found (see section on Metacognition). The second reason I find this statement interesting is because of Connor's explicit mention of the fact that "*the moment of 'seeing' it, i.e., of* *recognizing the pattern, is much more powerful than the moment of hearing or reading some fact*". Although I'm not certain what the 'it' that he is referring to, such a statement could speak both to the cognitive and the affective aspect of the AHA! experience. The cognitive dimension refers to the depth of the understanding that is gained and the affective dimension speaks to the emotions that are felt at '*seeing*' it.

Persistent Work

Persistence is a crucial element in the AHA! experience. Persistence is what describes the initiation phase of the AHA! experience. One must persist through this period of unresolved effort in order to move onto the incubation and eventual illumination phases of the AHA! experience. This contribution of persistence, although mentioned by many mathematicians, was best articulated in the comments of Henry McKean.

> *I don't believe that any true progress arises spontaneously. I believe it is always the result of lots of hard work, covert or overt, with the understanding that old work will sometimes come into a new focus so that you get something, if not for free, then at no extra cost. Such ``inspiration'' is the outcome of covert work and so can be surprising, but the work has to have been done, even if invisibly.*
>
> \- Henry McKean

> *Obviously you work like hell and once in while you notice something really unexpected.*
>
> \- David L. Donoho

> *Namely one has to spend much time on the subject before one gets inspiration.*
>
> \- Gerd Faltings

> *I found that in order to do creative work, I had to be at it without interruption for at least a week at a time.*
>
> - Peter J. Huber

> *In my principle discoveries I have always been thinking hard trying to understand some particular problem. Often it is just a hard slog, I go round arguments time and again seeking for a hole in my reasoning, or for some way to formulate the problem/structures I see. Gradually some insights builds and I get to "know" how things function.*
>
> - Dusa McDuff

The persistent effort can also occur after the initial inspiration as one works to verify and extend the idea. This was especially evident in the anecdote provided by Bombieri where he highlights both the perseverance as well as the obsessive intensity with which he applies it. Huber and Efron also offer very concise comments on the topic.

> *I worked three days and three nights never taking a rest save for eating a little and drinking coffee.*
>
> - Enrico Bombieri

> *Inspiration starts things, but only hard work really gets anywhere.*
>
> - Brad Efron

> *When I had a successful idea, I could not let loose and worked furiously.*
>
> - Peter J. Huber

However, persistence, like the AHA!, are also an important element of 'doing' mathematics in general as attested to by the following three mathematicians.

> *The main reason for occasional success is perseverance; never give up on a problem, continue day after day, week after week... also when it looks hopeless.*
>
> - Ulf Grenander

> *First, you can only do something worthwhile by devoting an embarrassingly extreme amount of time preparing yourself both within a specialty and by reading voraciously and very broadly*

outside the specialty as well. If they only knew the amount of dedicated concentrated effort involved, most people would be shocked and repelled at the sacrifice involved. (Of course, a few people will have exactly the kind of obsessive personality that drives them to this kind of effort, the rest would find it an unimaginable deprivation). I think this is true even of the greatest mathematicians, and I'll bet it is true of greatness in many other fields as well. [..] In olden times, where there was a heavier emphasis on gruelling hard work in school, the tendency to "see flashes" was probably very muted. So that the concept of 'genius' that was so revered previously is, I think, simply people being jealous of what they do not have. I would not valorize something simply because there was a "flash".

- David L. Donoho

It requires persistence and ability of high orders which are rare separately.

- Dick Askey

Deriving and Re-Creation

A theme that emerged out of questions three and four, but also resonated with comments from responses to other questions, was the importance of deriving mathematics for oneself. In part, this theme reflected the personal practices of assimilating the work of others through active re-creation of the mathematics rather than passive reading of it.

I find it very hard to learn the results of others these days unless they are very close to my own research interests. I used to be able to absorb things rather passively, just reading and doing over the chains of ideas. Now I need to work out examples, specific instances of the new ideas to feel that I have any real understanding. Anything one creates oneself is much more immediate and real and so harder to forget.

- Dusa McDuff

Learning work of others of course means following their thoughts, as opposed to thinking oneself. However I usually try whether I can find my own proofs for the assertions before I dig into their details.

- Gerd Faltings

> *When there is a need to fully assimilate something, I must redo everything in my own way.*
>
> - Benoit B. Mandelbrot

> *I've forgotten what Hadamard had to say on this, but for me there's no difference, - in order to `understand', I have to (re)create. To be sure, it's much easier to follow someone else's footsteps, i.e., it is much easier to prove a result one knows to be true than one that one merely guesses to be true.*
>
> - Connor

> *I have always studied results discovered by others, but frequently reworking the material in my own way. [..] I am impatient when reading work by others and usually try to work things out myself.*
>
> - Dick Askey

However, the theme also manifested itself as advice to young mathematicians about how to best approach doing mathematics in the expansive responses of both Kleitman and Huber.

> *You cannot tell when a successful idea comes to you whether it is luck that it did. On the other hand you can position yourself to be lucky by thinking hard about the problem, and by practicing. I do think that if someone wants to do this they should try to train their minds by exercising this skill on problems whose answer is known. That is, they should try to figure them out themselves. I try to train students to read a paper by first reading enough to find out what the author is trying to accomplish, and then put the paper down and try to think out an approach of your own to accomplishing it. If a student succeeds immediately in seeing what to do, the paper could not have been very good. More likely, he or she will fail; after thinking for a certain amount of time, one should go back to the paper and find a clue from it, and try again with this clue. This process can be iterated until the student can solve the problem. When I started on thesis research as a graduate student, my advisor gave me and a number of other prospective advisees a practice problem. Now it so happened that several weeks before I had heard a lecture about this very problem and read a paper in which it was solved. In giving the advisor my solution to this problem I of course made extensive reference to that paper. His reaction was: why did I want to look at a reference rather than*

> *trying to do the problem myself? I was too embarrassed to explain that I read the paper first; but I took his comment to heart and from then on have attempted to learn as much as I can myself, at least in part.*
>
> \- Dan J. Kleitman

> *If you have an idea, develop it on your own for, say, two months, and only then check whether the results are known. The reasons are: (1) If you try to check earlier, you won't recognize your idea in the disguise under which it appears in the literature. (2) If you read the literature too carefully beforehand, you will be diverted into the train of thought of the other author and stop exactly where he ran into an obstacle. This happened to a friend of mine, who started three Ph.D. theses in totally unrelated fields, before he finished one.*
>
> \- Peter J. Huber

This result speaks in support of such methods of instruction as discovery learning (Bruner, 1961; Dewey, 1916), constructivism (Steffe, Cobb, & von Glasersfeld, 1988), teaching through problem solving (Cobb, Wood, & Yackel, 1991), teaching through problem posing (Boaler, 1997; Brown & Walter, 1983), and the Moore Method of instruction (Jones, 1977) mentioned in the discussion of question four.

The Role of Significance

Several of the mathematicians mentioned significance as one of the characteristics that, both sets the AHA! experience apart from other mathematical experiences, and regulates the intensity of the experience. However, their usage of the term is problematic. Consider the following three excerpts:

> *It is, in my experience, just like other AHA! experiences where you suddenly "see the light". It is perhaps a little more profound in that you see that this is "important". I find that as one gets older, you*

> *learn to recognize these events more easily. When younger, you often don't realize the significance of such an event at the time.*
>
> - Jerry Marsden

> *What I think you mean by an AHA! experience comes at the moment when something mathematically significant falls into place.*
>
> - Wendell Fleming

> *The depth of the experience depends on how profound the ultimate result is.*
>
> - Michael Atiyah

Each of these three comments uses significance (or important, or profound) in a post-evaluative sense. That is, they speak of how significant a new idea will turn out to be once verified. However, such a thing cannot be known in the instance of the AHA!. It is only through verification, a potentially lengthy process, that this can be truly ascertained. If age has an effect on this, as Marsden claims, it would only serve to shorten the process of verification, not eliminate it as would be necessary for the significance to truly be known in the instance of the AHA! This is particularly pertinent to Fleming's comment that the AHA! occurs "*when something mathematically significant falls into place*".

However, such statements cannot be ignored. In order for the mathematicians to be associating the AHA! experience with the significance of the idea that reveals itself at the time of illumination then one of three things must be occurring. First, they are consciously suspending any evaluation of an experience as to whether or not it is an AHA! experience until after all the results have been checked. This may, in fact, be what Atiyah is doing in ascertaining the depth of the experience. Secondly, their recollection of the experience is being influenced by the outcome of the eventual evaluation of the idea that was presented to them during their AHA! experience. In psychology this is referred to as memory reconstruction (Whittlesea, 1993) and has a large amount of empirical data to support it. The third option, and the one that I find most likely, is that although the absolute significance of a find may ultimately be verified, at the time of the AHA! experience what they are, in fact, experiencing is a *sense* of significance. This *sense* of significance is not too dissimilar from the *sense* of certainty that they also experience, and like certainty may in the end prove to be unfounded. This possibility is displayed in Fleming's complete passage.

> *What I think you mean by an AHA! experience comes at the moment when something mathematically significant falls into place. This is a moment of excitement and joy, but also apprehension until the new idea is checked out to verify that all the necessary details of the argument are indeed correct.*
>
> \- Wendell Fleming

Although Fleming begins the passage with an absolute usage of significance, stating that the AHA! occurs when something "*mathematically significant falls into place*", he softens this stance in his second sentence where he acknowledges that he is filled with a feeling of apprehension until the results are verified. Fleming's statement clearly indicates that regardless of how he uses the term *significance*, he sees it as temporary and tentative. The same theme is also nicely demonstrated by Huber's response to question six where he makes a distinction between the AHA! experience and the EUREKA! experience.

> *First I would have a promising, brilliant(?) idea (the AHA! event) which would induce me to drill. But the EUREKA event ("I found it!") at best would come hours or days later, if and when the oil would begin to gush forth. That the idea had been brilliant and not merely foolish would be clear only in retrospect, after attempts to verify and confirm it. And later on one tends to suppress and forget foolish*

ideas because they are embarrassing (but they are indispensable companions to the brilliant ones!).

- Peter J. Huber

The Contribution of Luck

As mentioned earlier in the chapter there are two types of luck, *intrinsic chance* and *extrinsic chance*. Intrinsic chance deals with the luck of coming up with an answer, of having the right combination of ideas join within your mind to produce a new insight. This was discussed by Hadamard (1945) as well as by a host of others under the name of "*the chance hypothesis*". Extrinsic chance, on the other hand, deals with the luck associated with a chance reading of an article, a chance encounter, or a some other chance encounter with a piece of mathematical knowledge, any of which contribute to the eventual resolution of the problem that one is working on. This is best demonstrated by the words of Mandelbrot.

> *Do I experience feelings of illumination? Rarely, except in connection with chance, whose offerings I treasure. In my wandering life between concrete fields and problems, chance is continually important in two ways. A chance reading or encounter has often brought an awareness of existing mathematical tools that were new to me and allowed me to return to old problems I was previously obliged to leave aside. In other cases, a chance encounter suggested that old tools could have new uses that helped them expand.*
>
> - Benoit B. Mandelbrot

Until my work with the mathematicians I had not ever considered the role of this second type of chance in the mathematical experience in general, and in the AHA! experience in particular. Clearly, extrinsic chance is an important contributor to both types of experiences and needs to be considered when attempting to facilitate such experiences in the classroom. This is not a trivial task, however. After all, how do you integrate *luck* into your teaching practices? The answer to this, I believe, can be found in the words of Jerry Marsden.

> *As my anecdote indicates, it is not just chance, but rather inspiration in the presence of lots of surrounding information. The surrounding information is really crucial, I believe.*
>
> *- Jerry Marsden*

An environment needs to be created in which there is lots of "*surrounding information*". This makes sense, as more information increases the chance that any one individual is going to 'bump' into the one piece of information that they need in order to solve their problem.

The De-Emphasis of Details

Another theme that emerged from this study was the role that detail does NOT play in the learning of mathematics. Many of the participants mentioned how difficult it is to learn mathematics by attending to the details, and how much easier it is if the details are de-emphasized. Solomon Feferman nicely captures this idea in his comment.

> *Understanding others is often a painful process until one suddenly goes beyond the details and sees whole what's going on.*
>
> - Solomon Feferman

In some cases, this also manifested itself as a strategy for problem solving and research.

> *Get the basics of the problem firmly and thoroughly into the head. After that, an hour or two each day of thinking on it is all that's*

> *needed for progress. [..] For that reason, I started some 20 years ago to ask students (and colleagues) wanting to tell me some piece of mathematics to tell me directly, perhaps with some gestures, but certainly without the aid of a blackboard. While that can be challenging, it will, if successful, put the problem more firmly and cleanly into the head, hence increases the chances for understanding. I am also now more aware of the fact that explaining problem and progress to someone else is beneficial; I am guessing that it forces one to have the problem more clearly and cleanly in one's head.*
>
> \- Connor

In presenting his strategy for getting " *the basics of the problem firmly and thoroughly into*" his head, Connor has come up with a strategy that de-emphasizes the details by forcing the transmission of the problem through a medium wherein details are impossible. In his comment, he also introduces a subtly different role for talking that I alluded to in the previous section. More than simply moving information around, talking de-emphasizes details and, as a result, will "*put the problem more firmly and cleanly into the head, hence increases the chances for understanding*." This has pedagogic implications that are not so obvious and will be discussed in greater details in the next chapter.

The Role of Talking

In addition to surrounding information, there also needs to be a mechanism by which this information is transmitted. One such mechanism is talking. It is clear from the mathematicians' responses that they have a much higher regard for transmission of mathematical knowledge through talking than through reading. This is best summarized in the comments of Marsden and Papanicolaou.

I assimilate the work of others best through personal contact and being able to question them directly. [..] In this question and answer mode, I often get good ideas too. In this sense, the two modes are almost indistinguishable.

- Jerry Marsden

I get most of my real mathematical input live, from (good) lectures or one-on-one discussions. I think most mathematicians do.

- George Papanicolaou

Context of AHA! Experiences

Moments of illumination and insight are purported in the literature to occur in an untimely fashion. That is, they happen during times of non-mathematical activities such as bathing, walking, and sleeping. Relevant anecdotes spread throughout this chapter support this untimely occurrence of AHA! experiences reporting instances of illumination while showering, walking, sleeping, talking, cooking, driving, eating, waking, and riding the subway. To accentuate this I gather together these anecdotes here.

This can occur while I am officially working. But it can also occur while I am doing something else, having a shower, doing the cooking. I remember that the first time I felt creative in math was when I was a student (undergrad) trying to find an example to illustrate some type of behavior. I'd worked on it all the previous evening with no luck. The answer came in a flash, unexpectedly, while I was showering the next morning. I saw a picture of the solution, right there, waiting to be described.

- Dusa McDuff

While at a meeting in Philadelphia, I woke up one morning with the right idea.

- Dick Askey

And relevant ideas do pop up in your mind when you are taking a shower, and can pop up as well even when you are sleeping, *(many of these ideas turn out not to work very well) or even when you are driving.*

- Dan J. Kleitman

In regard to illumination, I would like to add that in my case the best instances have been at night when I am laying in bed, somewhere between consciousness and sleep.

- Demetrios Christodoulou

I distinctly remember the moment early in our collaboration when I saw how to get past one of the major technical difficulties. This happened while walking across campus after teaching a class.

- Wendell Fleming

It may have been in the shower that it just occurred to me that the work of some of the classical authors could be generalized in a certain way [..] I can be talking to a colleague or my wife or eating breakfast and suddenly, like a voice from the blue, I get told what to do.

- Jerry Marsden

The overt work is much the same as it always was. The covert work (in bed, on the subway, in dreams) is harder now.

- Henry McKean

I'm convinced that I do my best work while asleep. The evidence for this is that I often wake up with the solution to a problem, or at least with a clear idea of how to proceed to solve it.

- Charles Peskin

Profound Comments

This last section is not so much based on a theme as a collection of leftover comments that I felt were too profound to discard. They reflect the personal philosophies and deeply rooted beliefs of these mathematicians. They are succinct, self contained, and inspirational in their own right. As such, I see no need beyond this introduction to comment on them.

Bye and bye I see it, often quite suddenly, and realize that it's all quite simple, as mathematics properly understood must always be.

- Henry McKean

I was not going after it – it just happened. This is the opposite of the view of our current grant system, that imagines that one knows in advance what you want to do and then you go out and do it – like building a house. Research, really good research is not like that at all.

- Jerry Marsden

For me, and in this I am inspired by Grothendieck, the ideal is a proof which is trivial, because it has been preceded by the "correct" definitions.

- Pierre Rene Deligne

My attitude to mathematics has changed radically since I was a student. At that time I though of mathematics as a body of theorems, a static concept. I learned later to look at it as a problem solving activity. The theorems are still important, but perhaps less so nowadays.

- Ulf Grenander

At the level of words, there are really no new ideas. Good results do not come from inventing new words. And even at a somewhat higher level there are not really new ideas. It is extended combinations of ideas that can be new and can solve difficult problems.

- Dan J. Kleitman

Education has nothing to do with this. It has to do with NOT becoming deeply attached to some topic or topics but developing an all purpose methodology that can be applied to topics cross subject. And working diligently and in an organized away.

- David L. Donoho

Behind beautiful and seemingly important formulas there must be deeper ideas.

- Dick Askey

Conclusions

When Hadamard embarked on the research that eventually led to the writing and publication of his book he hoped to unlock some of the mysteries of the secret workings of the unconscious mind. When I resurrected his survey, however, I did so for the purposes of explicating some of the elements of the phenomenon itself. That is, I wanted to establish what it was about the AHA! experience that sets it apart from other mathematical experiences. With regards to this issue I have come to several conclusions.

First and foremost, my initial understanding of the essence of AHA! experience has been extended with regards to the role of the *suddenness, certainty*, and *significance* with which a solution presents itself. As presented in chapter one, the literature on invention and discovery treats the first two of these as defining characteristics of the phenomenon; suddenness has been used to describe the rapid appearance of an insight, while certainty describes the accompanying sense that the insight is correct. In this study *significance* joins the ranks of these defining characteristics as a second feeling that accompanies the sudden appearance of an insight. That is, illumination is marked both by a *sense* of certainty AND a *sense* of significance. It should be noted, however, that these senses of certainty and significance are just that, senses. In the immediacy and brevity of an AHA! experience neither the certainty nor the significance of an insight can be checked.

Suddenness, itself, also took on greater meaning through the interpretation of the data in this study. Classically, the characteristic of suddenness has been used in decontextualized isolation as a descriptor of *rapid* appearance of a solution. This is the case in the literature, and this was the case with some of the responses in this study. However, it was also the case that many of the responses were embedded in the larger context of the AHA! experience. From these cases of contextualized suddenness it becomes clear that the feeling is not always produced by the rapidity with which an idea appears, but on the untimely nature of its appearance. That is, suddenness is defined by what the person is doing when the solution appears. Or rather, what the person is not doing (i.e. mathematics). Examples of showering, walking, sleeping, talking, cooking, driving, eating, waking, and riding the subway appear within the anecdotal data and speak to the contextual dependency of the feeling of suddenness. This makes sense in light of the work by Hadamard (1945) wherein he uses the analogy of speech[20] to demonstrate the rapid and effortless way in which most ideas come to mind. As such, rapidity is not unique to moments of illumination, and as a result can neither punctuate nor accentuate the feeling of suddenness that accompanies AHA! experiences.

What remains is that the defining characteristics of an AHA! experience are dependent on the WAY in which the idea comes, rather than on the idea itself. As a result, this is also what sets the AHA! experience apart from other mathematical experiences. This is nicely summarized by Connor.

> *When, after some considerable, quite non-productive effort, usually while not at all consciously working on the problem, there appears,*

[20] Hadamard (1945) purports the existence of what he calls the antechamber of the mind in his explanation of the effortless, seamless, and rapid way in which the next word or next sentence comes to mind during a conversation.

for no apparent reason, in your brain the answer to that problem – that's the AHA! experience.

- Connor

This conclusion is further supported by the mathematicians' anecdotal descriptions of AHA! experiences, not one of which depends on the details of the idea behind it, as can be seen in Dusa McDuff's comments.

I remember that the first time I felt creative in math was when I was a student (undergrad) trying to find an example to illustrate some type of behaviour. I'd worked on it all the previous evening with no luck. The answer came in a flash, unexpectedly, while I was showering the next morning. I saw a picture of the solution, right there, waiting to be described.

- Dusa McDuff

My conclusions from the last chapter regarding the affective component of the AHA! experience have also been confirmed. In chapter five it was concluded that not only is the AHA! accompanied by an emotional response, but that that response is substantial enough to alter the negative belief structures and poor attitudes of resistant mathematics students. Poor belief structures and bad attitudes were not an issue with the participants of this study, but that did not eliminate the emotive response that they had during an AHA! experience as expressed in the comments by Huber and Connor.

When things had been settled and written up, I felt exhausted and empty, and itched until I had a new promising idea.

- Peter J. Huber

Certainly it is standard among mathematicians to enjoy these AHA! moments while they last and postpone for a bit (e.g., until the next day) the necessary checking of the insight.

- Connor

Likewise, my understanding of how to invoke AHA! experiences, which until this point had been almost non-existent, has been greatly enhanced by the anecdotal accounts of the mathematicians in this study. Although these ideas are not conclusive, the emergent themes regarding the contribution of luck, the role of talking, the de-emphasis of details, the importance of re-creation, and the role of perseverance have informed me of some of the environmental elements that may contribute to the occurrence of AHA! experiences. How these elements can play out in the context of a classroom is discussed in detail in the next chapter.

CHAPTER SEVEN
OCCASIONING AHA! EXPERIENCES: THE CASE OF PRESERVICE TEACHERS

In chapter four I presented my initial thoughts on the possibility of controlling the AHA! experience in a clinical setting through the careful orchestration of the mathematical stimuli that a student receives. As it turned out this approach was problematic at a number of different levels, not the least of which was the inadequacy of the clinical setting as an environment conducive to the AHA! experience. In my work with the mathematicians, presented in chapter six, I came to see that the occurrence of an AHA! experience was often precipitated by a chance encounter and supported by talking, the de-emphasis of details, a focus on re-creating mathematics, and perseverance. Given the fleeting quality of many of these supporting elements, as well as the elusive nature of chance, it became clear that the best that could be done, with regard to controlling the occurrence of an AHA! experience, would be to create an environment conducive to the *occasioning* of an AHA! experience. That is, increase the likelihood that one would happen.

In this chapter I present the results of a study in which the aforementioned contributory elements for the occasioning of AHA! experiences were used to restructure the delivery of two *Designs for Teaching Mathematics* courses offered at Simon Fraser University. In particular, I reshaped the course environment in such a way as to increase the likelihood of an AHA! occurring. These occurrences were then tracked through a new method of journaling that was created specifically for capturing such phenomena in a sincere and accurate manner. Because this was a new, and untested, form of journaling a portion of the study was designed to validate the effectiveness of the journals while simultaneously making use of the rich set of data that they provide. In this way, this particular study delivers a greater understanding of the AHA! experience as well as a new methodology for collecting data on the experience.

METHODOLOGY

The two courses where I implemented my restructured instructional design were *Education 415: Designs for Learning Secondary Mathematics* and *Education 475: Designs for Learning Elementary Mathematics*. These are courses specifically designed to provide preservice teachers with the foundational knowledge and skills requisite to teach mathematics at their respective levels. As such, the focus of the individual courses is more on teaching than on the specifics of mathematics content, although such content is often used as the context in which teaching and learning are discussed.

At the time of the study I was teaching these two courses concurrently. The courses are offered for 13 weeks and each class met once each week for four hours. Because of the nature of the courses, and the timing with which they are offered the majority of the students who take these courses were enrolled in the teacher preparation program offered at the university.

For reasons of clarity the students enrolled in Education 415 will from now on be referred to as *secondary participants*. Likewise, the students enrolled in Education 475 will be referred to as *elementary participants*. Although the terms secondary and elementary have traditional meanings, I have chosen to refer to these two groups of preservice teachers in this manner because it is both concise and descriptive.

The Secondary Participants

Education 415 is usually offered once a year during the summer semester. In the year the study was conducted there were 34 students enrolled (18 males, 16 females), all but two of which were preservice teachers. The other two were practicing teachers who were taking the course as a prerequisite for entry into the Secondary School Mathematics Education master's program. A loosely conducted survey revealed that about half of the students were mathematics specialists, most of whom had completed an undergraduate degree with a major in mathematics. They saw themselves as capable of teaching mathematics up to the grade 12 level, including Calculus. Many of these students also saw themselves teaching junior sciences either out of interest or out of the realization that such was the reality for beginning teachers. The remaining half of the students considered themselves science specialists. They were enrolled in the course because they knew that it was likely they would be given teaching assignments that included mathematics. They did not see themselves teaching mathematics above the grade 10 level.

My experience teaching this course has been that the students enrolled in it are somewhat overconfident in their ability to teach. They have subject matter knowledge, whether it is in mathematics or science and they tend to assume that this means they can teach. Of course this is a generalization, but in comparing these students to the students encountered in teaching Education 475 it is an accurate one. This is not to say that the secondary participants do not accept ideas regarding teaching and learning, but only that they are more resistant to it than their elementary counterpart. They do, however, see the course as a necessary *hoop* to jump through in order to enter into the teaching profession.

The Elementary Participants

Unlike Education 415, Education 475 is offered several times over the course of the year, in several different formats. It is offered at least twice a year in "correspondence" formats by the Centre for Distance Education and in the summer there are usually three or four sections offered on campus. In the section that I was teaching there were 38 students enrolled (5 males, 33 females), all of whom were enrolled in the teacher preparation program. All but three of these 38 students considered themselves to be very weak in mathematics. However, they all knew that teaching mathematics was an eventuality in their chosen profession and had resigned themselves to the fact that they needed some help in developing the skills to do so.

My experiences in teaching this course had been nothing but positive. As apprehensive and fearful of mathematics as these students are, they are extremely open to, and appreciative of, any ideas that may help them to become better mathematics teachers. They are willing to engage in discussions regarding pedagogy and philosophy and are willing to reflect on how to apply these in their teaching.

Overview of the Courses

One of the biggest challenges in teaching these courses is getting the students to redefine their understanding of what mathematics is and what it means to teach and learn mathematics. Ideas regarding these issues are, for many, based solely on personal experiences as mathematics students in traditionally taught settings. As such, there is a tendency to see mathematics as a collection of facts to be memorized and skills to be mastered, and to see teaching mathematics as making this process as painless (and fun) as possible. These are crucial misunderstandings that need to be resolved before the specifics of teaching can begin to be dealt with. Such being the case, in the past I have implemented a framework of instruction in which the students are immersed in a mathematical experience that is, for many, completely new. There are many dimensions to this immersion, from group work to whole class discussions, but at the centre of it all is a heavy focus on mathematical problem solving.

Inspired by the work of Deborah Ball (Ball, 1997, 1996; Mosenthal & Ball, 1992; Price & Ball, 1998), for the first six weeks of the course I have the students constantly solving problems of varying difficulty. There are in-class problems, weekly homework problems, and a larger ongoing term problem. However, the solution of these problems is secondary to the engagement in the process of solving them. I want these students to see what it really means to 'do' mathematics. Although this approach is consistent with the ideas of discovery learning (Bruner, 1961; Dewey, 1916) and constructivist theories of learning (Steffe, Cobb, & von Glasersfeld, 1988), the goal is altogether different. Concerns with the students learning any particular mathematical content from their work on the problems are displaced by the need for them to see what it means to do mathematics in an environment where the process is valued more than the outcome and where trying and failing, and trying again, is preferable to being told how to do something. I want to provide some positive mathematical experiences for my more anxious, phobic, and resistant students and I want all of my students to see what it means to feel safe in doing mathematics. I wish for them to see what learning mathematics can look like, and through this I hope to affect their perceptions of what it could mean to teach mathematics.

Other aspects of the course, less relevant to the forthcoming discussion include the explicit attention paid to specific mathematics curriculum items as well as more global concepts pertaining to assessment, group work, lesson and unit planning, didactics, and pedagogy.

The in-class work is accompanied by the use of reflective journals in which the students respond to critical questions posed at the end of each session. Through these journals the students make explicit for themselves their thoughts on issues in the teaching and learning of mathematics, and are able to challenge and refine these thoughts. Prevalent among the critical questions used are: *What is mathematics? What does it mean to learn mathematics? What does it mean to teach mathematics?* These are asked on three different occasions – at the beginning of the course, midway through the course, and at the end of the course – and act as markers in the evolution of their thoughts on these issues.

Because this method has been successful in the past I decided to use it as the template for the courses that were to be used for this study. The few changes that were made were inspired by some of the results from my work with the mathematicians, presented in chapter six. In particular, the way in which problem solving was incorporated into the course was restructured. To begin with there was greater emphasis placed on talking and less emphasis placed on details. This manifested itself primarily in the way the problems were introduced. Instead of providing the students with precisely worded instructions and explanations of the problems, as had been done in the past, the problems were introduced orally, perhaps with some demonstrations and hand gestures. This focus on the oral delivery of problems was problematic, especially for students who were absent, but it was made clear that it was expected and that they should contact a peer for instructions should they miss a session. Students were allowed to take notes if they wished, but were encouraged to do so only after they had played around with the problem.

Of course, there were a few exceptions to this. If the details of the problem were so intricate that they could not be remembered the problem was provided in writing (see for example *The Treasure of Captain Bird* in the section on Tasks). Also, if there were details that were beyond the scope of getting into the problem then written instructions of those details were also provided (see for example *The Card Array Problem* in the section on Tasks). Finally, in the case of the term problems (see *Bit String* and *Pentominoe Problem*s in the section on Tasks), written instructions of both the problem and the details were provided.

This focus on talking also played a role in the use of group work. Group work is something that I have always valued and had always used. However, given the mathematicians' comments regarding talking I decided to place even greater emphasis on group discussion and peer interaction. One way in which this was accomplished was through the increased provision of class time to work together on problems. An altogether different way of facilitating interaction, however, was to physically remove myself from any peer discussions centred on the problems that had been given out. I knew from past experience that my presence changed the nature of conversations from hypothesising to questioning. That is, the students would stop talking to each other and start asking me to validate their claims. As such, an explicit effort was made to remove myself from peer interactions as soon as such requests for validation arose, sometimes even leaving the room. Of course, this made observation of any AHA! experiences extremely difficult, but I was confident that the use of problem solving journals would capture any such experiences for later analysis.

Another adjustment that was made had to do with time. This was primarily in the form of much more class time to work on problems and came in two forms:

time immediately after being assigned the problem, and time to revisit already assigned problems. As already mentioned, both of these allocations were provided in order to create opportunities for discussion. However, they also served the purpose of creating an interval like re-visitation of problems mentioned by the mathematicians. One further use of time was incorporated; deadlines were extended as much as possible. That is, I allowed much more time to work on problems than I had ever done in the past. This is in keeping with the idea that problem solving can take an extremely large amount of time, and to be done well, an extremely large amount of time should be provided.

The final change made in the delivery of the course has to do with surrounding information. In each of the courses I experimented with this idea by filling the air with relevant, but not explicitly linked, information. The results of both of these experiments will be discussed in the *Results* section of this chapter.

The Tasks

In what follows I present some of the problems that were used in the two courses. These problems are sorted by course and by the way in which they were used (in-class problem, homework problem, term problem).

As already mentioned, these problems were given primarily as oral instructions. For obvious reasons I present them here in their written form. However, as also mentioned, there were exceptions to this oral delivery of problems, either in part or in whole. In order to differentiate between the two forms I introduce a notational convention here. Any problems, or parts of problems presented in italics represent information that was conveyed orally through discussion and/or presentation. For brevity, these passages have been condensed into more precise language. Also of note is the fact that many of these problems were presented to the students without names. Names are provided here for the purposes of reference.

Education 415: Designs for Teaching Secondary Mathematics

- Fibonacci String (in-class problem)

  Create a sequence of numbers as follows. Pick any whole numbers to be term #1 and term #2. These are called the seed numbers. Term three will be the sum of term #1 and term #2. Term four will be the sum of the two previous terms, and so on. I'm interested in term #5. Find all the seed numbers such that term #5 will be 100.

- Student Moves (in-class problem)

  The desks of 25 students are arranged in a 5 x 5 array. The teacher comes in and tells the students that everyone has to change to a different desk. The stipulation is, however, that they are only allowed to move into an adjacent seat (not kitty-corner) and everyone must move. Can it be done?

- Chessboard and Dominoes (in-class problem)

  A chessboard has two squares removed, one each from two opposite corners. Assuming that a domino placed on the chessboard covers exactly two squares, can the modified chessboard be tiled with dominoes?

- The Treasure of Captain Bird (homework problem)

  The treasure of Captain Bird is buried on the island of the parrot. Near the centre of this island three great trees form a triangle. The mightiest of the three is a great oak older than the treasure itself. Towards the west of the oak, some distance away there stands an elm tree, and towards the east of the oak there stands an ash. To find the treasure of Captain Bird count out the paces from the oak to the elm. When you get to the elm make a precise left turn and count out the same number of paces. Mark this spot with a flag. Return to the oak and count out the paces from the oak to the ash. When you get to the ash make a precise right turn and count out the same number of paces. Mark this spot with a flag. The treasure lays buried midway between the two flags.

You rent a boat and set out for the island. When you get there, however, you discover that the oak tree is missing without a trace. Where is the treasure? Why is it there?

- Corner to Corner (homework problem)

  An NxN array has a red marker in every cell except for two. One corner of the array is left empty and the corner furthest from this empty cell has a white marker in it. What is the smallest number of moves required to get the white marker into the initially empty cell given that the only valid move is to move a marker into an adjacent empty cell (that is, not kitty-corner)?

- The Giant Wheel (homework problem)

  A very large wheel (diameter of 100 km or so) rolls slowly past your ground floor window. How is the light blocked out as it passes your window (straight down, straight sideways, or on an angle) and is it dark for a long time or a short time?

- Card Array (homework problem)

  Take an entire deck of cards and lay them out in some sort of an array. Pick any card in the first row and allow that to be C_1. The numeric value of the card (face cards are valued at 1) tells you how far to count to get to C_2. Repeat this procedure, moving through the array in any agreed upon fashion (that is left to right, or snaking, etc.) until you get to the last card that you can got to without exceeding the array. Regardless of which card you start with, you will always end up on the same card.

  This doesn't always work. However, it almost always does. Why does it almost always work, and what has to happen for it not to work?

- Four Pockets (homework problem)

  A round table has four deep pockets equally spaced around its perimeter. There is a cup in each pocket oriented either up or down, but you cannot see which. The goal of the game is to get all the cups up or all the cups down. You do this by reaching into any two pockets, feeling the orientation of the glasses, and then doing something with them (you can flip one, two, or none). However, as soon as you take your hands out of the pockets the table spins in such a way that you can't keep track of where the pockets you have visited are. If the four glasses ever get oriented all up or all down a bell rings to signal you are done. Can you guarantee that you will get the bell to ring in a finite number of moves, and if so, how many?

- Bit String #1 (term problem option #1)

  Consider a string of 1's and 0's. Chunk this string into pairs starting at the left and then evaluate each pair. If a pair matches replace the pair with a 0, if it doesn't match replace it with a 1. If the string has an odd length, then consider the last bit as being unmatched. Repeat this procedure for the new string, etc. until there is only one bit left. Can you find a way to predict what the final bit will be based on the original string?

Consider the following string: **1 1 0 1 0 0 1 1 1 1 0 0 1 0 1**

<table>
<tr><td>1</td><td>1</td><td>0</td><td>1</td><td>0</td><td>0</td><td>1</td><td>1</td><td>1</td><td>1</td><td>0</td><td>0</td><td>1</td><td>0</td><td>1</td><td></td></tr>
<tr><td colspan="2">0</td><td colspan="2">1</td><td colspan="2">0</td><td colspan="2">0</td><td colspan="2">0</td><td colspan="2">0</td><td colspan="2">1</td><td colspan="2">1</td></tr>
<tr><td colspan="4">1</td><td colspan="4">0</td><td colspan="4">0</td><td colspan="4">0</td></tr>
<tr><td colspan="8">1</td><td colspan="8">0</td></tr>
<tr><td colspan="16">1</td></tr>
</table>

- Bit String #2 (term problem option #2)

  Consider a string of 1's and 0's. Chunk this string into adjacent pairs starting at the left and then evaluate each pair. If a pair matches replace the pair with a 0, if it doesn't match replace it with a 1. Repeat this procedure for the new string, etc. until there is only one bit left. Can you find a way to predict what the final bit will be based on the original string?

  Consider the following string: **1 1 0 1 0 0 1 1 1 1 0 0 1 0 1**

  1 1 0 1 0 0 1 1 1 1 0 0 1 0 1
  0 1 1 1 0 1 0 0 0 1 0 1 1 1
  1 0 0 1 1 1 0 0 1 1 1 0 0
  1 0 1 0 0 1 0 1 0 0 1 0
  1 1 1 0 1 1 1 1 0 1 1
  0 0 1 1 0 0 0 1 1 0
  0 1 0 1 0 0 1 0 1
  1 1 1 1 0 1 1 1
  0 0 0 1 1 0 0
  0 0 1 0 1 0
  0 1 1 1 1
  1 0 0 0
  1 0 0
  1 0
  1

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- Nim (in-class problem)

  The game of Nim is played between two people. Seven markers are placed in a row. Each turn you are allowed to remove one or two markers from the row. The player who removes the last marker (either on its own or as part of two) wins. What is the strategy for winning the game for any number of markers?

- Number Line (in-class problem)

  Create a number line with 0 at one end and 1000000 at the other. Where will 1000 be on the number line?

- Fibonacci String (in-class problem)

  As presented above.

- Magic Addition (homework problem)

  Write down any three 3-digit numbers as if you are going to add them up using column addition. I will then add two 3-digit numbers to this list and instantly tell you what the sum is. For example: you provide the numbers 271, 742, and 836. I will provide the numbers 728 and 257 and instantly tell you that the sum of all five is 2834. How do I do it so quickly?

- Magic Division (homework problem)

  Write down a three-digit number twice to make a six-digit number. I'll bet that this six-digit number is divisible by 7, 11, and 13. Why does it work? Show me how you know.

- Cats and Rats (homework problem)

  Lewis Carol posed the following problem: If 6 cats can kill 6 rats in 6 minutes, how many cats are required to kill 100 rats in 50 minutes?

- The Treasure of Captain Bird (homework problem)

  As presented above, with the exception that there was no requirement to explain why the treasure was where it was.

- Tea Party (homework problem)

  There is a mother-daughter tea that will be attended by 20 mother-daughter pairs, including the hosts. The rules of conduct are very strict; the host mother-daughter will greet everyone, all the guest mothers will greet everyone, and all the guest daughters will greet all the mothers only. How many greetings will there be?

- Pentominoe (term problem)

A pentominoe is a shape that is created by the joining of five squares such that every square touches at least one other square along a full edge. How many are there? Name them. If a Pentominoe is placed on the number grid will the sum of the numbers it covers up be divisible by 5? When will it? When will it not? If not, what will the remainder be? Why?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Capturing AHA! Experiences: A New Form of Journaling

In conjunction with the use of problems I had also, in the past, made use of problem solving journals. In these the students were to record their efforts, successes, failures, frustrations, ideas, and emotions as they engaged in solving assigned problems. These journals had been used for several reasons. First of all, it served to model for the preservice teachers an alternate form of assessment that honoured the problem solving process, a facet of 'doing' mathematics that was being emphasised for them. Equally important, as a form of data collection, it had the potential to provide insights into student thinking. Unfortunately, as successful as these problem solving journals had been in displaying the value of the problem solving process, they had been equally unsuccessful at providing any real insight into students' thinking processes. Their ability to journal in a sincere and accurate fashion was too poor.

Typically, the secondary participants had produced the worst journals. It was extremely difficult for them to overcome their years of schooling and produce anything but a mathematical explanations of their solutions interspersed with a few sentences of language explaining why their mathematical solution or discovery was true. Their journal entries were almost completely devoid of any wrong turns in their problem solving process, no false conjectures, no wild goose chases; just the answer. Their work was reflective of the comments of one of the mathematicians surveyed for the study presented in chapter six.

> *In working on this problem and in general, mathematicians wander in a fog not knowing what approach or idea will work, or if indeed any idea will, until by good luck, perhaps some novel ideas, perhaps some old approaches, conquer the problem. Mathematicians, in short, are typically somewhat lost and bewildered most of the time that they are working on a problem. Once they find solutions, they also have the task of checking that their ideas really work, and that of writing them up, but these are routine, unless (as often happens) they uncover minor errors and imperfections that produce more fog and require more work. What mathematicians write, however, bears little resemblance to what they do: they are like people lost in mazes who only describe their escape routes never their travails inside.*
>
> \- Dan J. Kleitman

In such an account there is little concession given to the process of solving the problem, and none given to presenting the process. Such an attitude is likely due to a mathematical convention that has come to rely solely on deductive logic and proof as the only acceptable method of presentation of mathematical knowledge. Through their exposure to such convention over the course of their undergraduate education the secondary participants had likely come to adopt this method of communication as well.

The elementary participants had generally been more truthful in their journaling. Their entries were more reflective of the process, even venturing to tell of their exploits *inside the maze*. However, their narrations were critically monotone and completely devoid of any of the emotions that accompany the highs and lows of the problems solving process. If I were to get any useful insights regarding the problem solving processes and the AHA! experiences of either of these groups of students then I needed to restructure their method of journaling.

Save for very few exceptions, literature that details mathematician's problem solving efforts is equally unrepresentative of the true process of 'doing' mathematics. One exception to this is an account written by Douglas R. Hofstadter (1996) that tells the story of a mathematical discovery with amazing sincerity. It is detailed and complete, from initiation to verification. It tells the story of being lost in a maze, searching for answers, and in a flash of insight, finding the path out. The process of coming to know is not lost through the process of coming to present what is known.

Perhaps the reason that the account is so different is that Hofstadter is not a professional mathematician. He is a college professor of cognitive science and computer science, and an adjunct professor of history and philosophy of science, philosophy, comparative literature, and psychology; best known for his book *Gödel, Escher, Bach: an Eternal Golden Braid* (1980) for which he won both the American Book Award and the Pulitzer Prize. Although Hofstadter is not a professional mathematician he does do mathematics at a very high level and has a particular passion for Euclidean geometry. He also happens to have a long-standing interest in creativity and consciousness. As such, he has a unique appreciation for tracking his own creative endeavours.

In analysing Hofstadter's account it becomes clear that one of the reasons that it is so sincere is because of the way in which he incorporates the use of three different voices, a trinity of personas, in telling his tale. As this usage is implicit within his writing this was not immediately obvious, and it was only after several readings that I became aware of it. I have come to name these personas the *narrator*, the *mathematician*, and the *participant*. Each of these personas contributes to the anecdotal account in a different way. The *narrator* moves the story along. As such, he often uses language that is rich in temporal phrases: 'and then', or 'I started'. He also fills in details of the non-mathematical variety seemingly for the purpose of providing context and engaging content. The *mathematician* is the persona that provides the reasoning and the rational underpinnings for why the mathematics behind the whole process is not only valid, but also worthy of discussion. Finally, the *participant* speaks in the voice of real time. This persona reveals the emotions and the thoughts that are occurring to Hofstadter as he is experiencing the phenomenon. Together these three personas and their respective voices provide the details I suggest is required to successfully capture an AHA! experience in its entirety.[21]

To demonstrate these personas, I present a portion of the chapter that contains within it all three voices. Before I do, however, it would be useful to introduce the general context of his mathematical encounter. At the time of writing the chapter, Hofstadter has only recently come to be impassioned with Euclidean geometry and had never been introduced to the Euler line of a triangle[22]. When he did learn about it, however, two things immediately struck him: the connectivity of seemingly different attributes, and the exclusion of the incentre[23]. So, he began a journey of trying to find a connection between the

[21] Although I do not explicitly draw a clear distinction between *persona* and *voice*, distinction does exist. Persona, or stance as it is sometimes called, is "the created personality put forth in the act of communicating" (Hyland, 2000, p. 101) while voice is "the speaking personality that is recognized, heard, or valued in an utterance or text in a particular context" (Maguire & Graves, 2001, p. 564). That is, the persona is the position, or perspective, from which a person speaks, while the voice is how they speak from this position.

[22] The Euler line of a triangle is a line that connects the *orthocentre* (the intersection of the altitudes of a triangle - H), the *centroid* (the intersection of the medians of a triangle - G), and the *circumcentre* (the intersection of a triangles perpendicual bisectors - O). See figure 4, below.

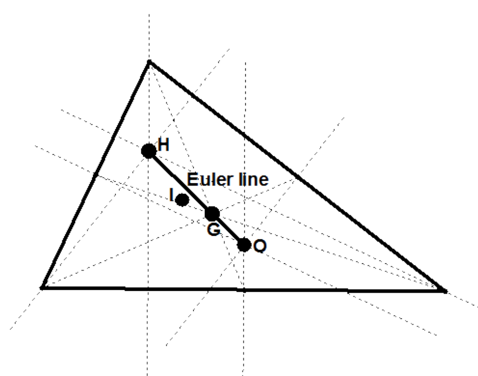

Figure 4: The Euler Line

[23] The intersection of a triangle's angle bisectors. See I on figure 4.

Euler line and the incentre. The result of this journey was the discovery of an existing line analogous to the Euler line. The symmetry between these lines helped him to formulate his attack and to then create a third line analogous to both these lines. Along the way he not only found connections between the incentre and the Euler line but also to a number of other known and unknown points of significance. At the point in the passage presented below Hofstadter has just discovered something about the incentre.

> *One day I made a little discovery of my own, which can be stated in the following picturesque way: If you are standing at the vertex and you swing your gaze from the circumference to the orthocentre, then, when your head has rotated exactly halfway between them, you will be staring at the incentre. More formally (see Figure 5), the bisector of the angle formed by two lines joining a given vertex with the circumcentre and with the orthocentre passes through the incentre. (A more technical way of characterizing this property is to say that O and H are "isogonic conjugates".) It wasn't too hard to prove this, luckily. This discovery, which I knew must be as old as the hills, was a relief to me, since it somehow put the incentre back in the same league as the points I felt it deserved to be playing with. Even so, it didn't seem to play nearly as "central" a role as I felt it merited, and I was still a bit disturbed by this imbalance, almost an injustice.*
>
> (Hofstadter, 1996, p. 4)

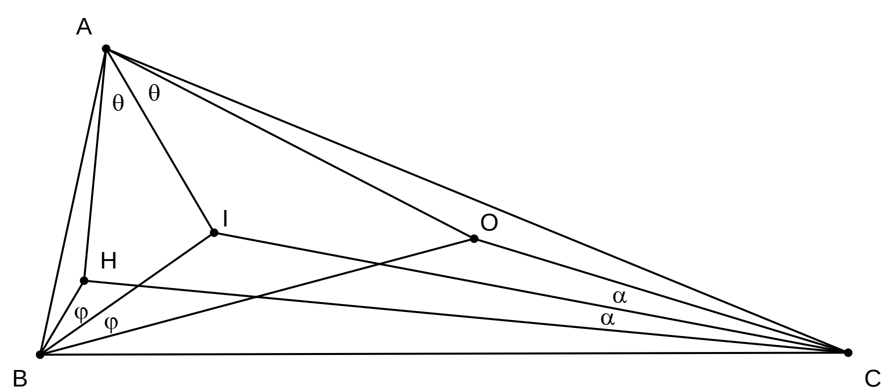

Figure 5: Triangle With Incentre, Orthocentre, and Circumcentre

Even from this brief excerpt it can be seen how the three personas interact with each other, while at the same time presenting different aspects of the mathematical experience. It begins with "One day …", a clear indicator that the *narrator* will be speaking.

> *One day I made a little discovery of my own, which can be stated in the following picturesque way: If you are standing at the vertex and you swing your gaze from the circumference to the orthocentre, then, when your head has rotated exactly halfway between them, you will be staring at the incentre.*

Hofstadter is telling us what he has found in an informal yet descriptive way. This is followed by his *mathematician* persona coming in and formalising this finding in a more precise and mathematical way.

> *More formally (see Figure 5), the bisector of the angle formed by two lines joining a given vertex with the circumcentre and with the orthocentre passes through the incentre. (A more technical way of characterizing this property is to say that O and H are "isogonic conjugates".) It wasn't too hard to prove this, luckily.*

Finally, the *participant* reveals how he feels about his finding and what thoughts this find is precipitating.

> *This discovery, which I knew must be as old as the hills, was a relief to me, since it somehow put the incentre back in the same league as the points I felt it deserved to be playing with. Even so, it didn't seem to play nearly as "central" a role as I felt it merited, and I was still a bit disturbed by this imbalance, almost an injustice.*

The interplay present in this passage is typical of the first six pages of the chapter. At that point in the account Hofstadter has his AHA!, which is revealed in his last use of the *participant's* voice. After this point there is a brief interplay between the *narrator* and the *mathematician* and then the voice of the *narrator* also disappears forever. The last seven pages of the chapter are comprised of the *mathematician* articulating and proving his discovery.

The most interesting thing about Hofstadter's use of these three voices is what it reveals about the type of journals that my students had produced in the past. The journals produced by the elementary and secondary participants in the past, at best, had been a combination of the voice of the *mathematician* and the *narrator*. I mentioned that such journals failed to represent the true thoughts and feelings that the students were experiencing during their mathematical experience. Although this had been frustrating I had failed to find a solution to it. Repeated urging to be truthful in their writing only resulted in more detailed narratives with more descriptions of failed attempts and mistaken assumptions. The third voice was missing, but until I read Hofstadter's account of his discovery I did not even know it could exist. In his description was the missing piece that was required to elevate the students' journaling to the level of detail that was needed to really see their mathematical thinking[24], and to capture their AHA! experiences.

I introduced this form of journaling to my students on the first day of the course. That is, I introduced each of the three personas and what their respective

[24] Andrew Waywood (1992) has done work on creating a developmental model of students' mathematical learning through journaling. In this work he identified three types of journaling within his subjects. They are *recount, summary*, and *dialogue*. *Recounting* is very similar to what I refer to as the voice of the *narrator* and *summarizing* is virtually identical to the voice of the *mathematician*. *Dialogue*, however, is only part of what I refer to as the voice of the *participant*. For Waywood, dialogue is the self-talk that goes on in the journals, through which ideas are revealed. He does not, however, stipulate that *dialogue* contains any expressions of emotions. Both of these characteristics, presentation of ideas and emotions, make up the voice of the *participant*.

roles were in documenting problem solving efforts. Although there was no mention in what proportions they were to use them, it was made clear that they were expected to incorporate each of these three voices in their problem solving journals. Four weeks into the course their problem solving journals were collected and one specific homework problem was critiqued (*Corner to Corner* for the secondary participants, and *Magic Sum* for the elementary participants). This was followed by an in-class formal review of the three personas, examples of their voices, and a review of the expectations regarding the use of the three voices. Other than these moments of instruction (totalling no more than 60 minutes) and the critique of the problem solving journals at the four-week mark, no further class time was devoted to this topic.

Data Sources

The data for this study comes from three sources. These are: observation of in-class problem solving work, selected excerpts from their reflective journals (as described in the section Overview of the Courses, above), and their problem solving journals. The first of these, observations of in-class problem solving work, was difficult in that I usually removed myself from peer-interactions. However, I did manage to observe a few instances of AHA! experiences, both up close and from a distance. These observations were recorded immediately after class and checked against the participants' own accounts as presented in their problem solving journal.

The majority of the data, however, comes from the participants' journals. The reflective journal was used exactly as it had been in the past and consisted of their responses to key prompts asked at the end of each session. In particular, the questions that were used for this study were assigned at the end of class in week 10 and week 11 of the course.

<u>WEEK 11 REFLECTIVE JOURNAL PROMPT</u>
At some point during your efforts to solve some of the problems given in this course, you may have had what is referred to as an AHA! experience[25] *where the answer, or a hint, or an idea suddenly came to you. Pick the most powerful (you may do more than one if you wish) of these and tell the story around it. In doing so make sure you address the following questions:*

- *What was the question you were working on and where were you stuck?*
- *What did the AHA! give you – what were you able to do with the idea that came to you?*
- *Why do you think that particular idea came to you at that time? What do you think caused this AHA!*
- *How did it make you feel?*

<u>Week 12 Reflective Journal Prompt</u>
Continuing on the theme from last class' writing journal assignment reflect on your ***own*** *problem solving process. In particular, respond to the following questions:*

- *How do you go about solving problems?*
- *Does it always work, if so how often? For which problems in this course did this process work? For which didn't it, and what was it about those problems that made it so it didn't work?*
- *What makes a good problem?*
- *Which problem was your favourite, why? Which problem was most memorable, why?*

The problem solving journal was used as described above to record their efforts, successes, failures, ideas, and emotions encountered while working on the homework problems as well as their term problem. They were to use the voices

[25] Until week 11 of the course there had been no usage of the term 'AHA! experience', nor had there been any discussion of the essence of the experience.

of the three personas to present, as accurately as possible, their problem solving process. These journals were due at the beginning of class in week 11.

RESULTS

Although both the reflective journals and the problem solving journals contained useful data in their own right, it was through the coordinated usage of these journals that the data gains greater meaning and greater validity. Consequently, these journals were analysed both separately and in conjunction with each other. In what follows I present the most prominent of the themes that emerged as a results of this analysis.

Reflections on AHA! Experiences

In the reflective journals of the 34 secondary participants there were 26 students who claimed to have had an AHA! experience while working either on an in-class problem, a homework problem, or the term problem. Of the eight who did not make such a claim, six explicitly stated that they had not had one in working on any of the problems during the course, and two did not respond to the prompt at all. Table 3 summarizes these responses along with a detailed summary of which problems students chose to write about.

Table 3: Sorting of Secondary Participants' AHA! Responses by Problem (n = 34)

Problem chosen to write on	**Quantity (n=34)**	**% of n**
Claim no AHA!	*6*	*18%*
No response to prompt	*2*	*6%*
Student Moves	1	3%
Four Pockets	6	18%
Corner to Corner	4	12%
The Treasure of Captain Bird	6	18%
The Giant Wheel	2	6%
Card Array	1	3%
Bit String #1 (term problem)	2	6%
Bit String #2 (term problem)	4	12%

Table 4 summarizes the responses of the 38 elementary participants.

Table 4: Sorting of Elementary Participants' AHA! Responses by Problem (n = 38)

Response/Problem	**Quantity**	**% of n**
Claim no AHA!	*1*	*3%*
No response to prompt	*1*	*3%*
Magic Division	4	11%
The Treasure of Captain Bird	3	8%
Fibonacci String	2	5 %
Magic Addition	4	11%
Cats and Rats	3	8%
Pentominoe (term problem)	20	53%

There are a number of differences that become obvious when these two tables are compared. The first is that a much higher percentage of elementary participants had AHA! experiences while working on the term problem in comparison to the secondary participants. Aside from this elementary term problem, no other problem stood out in terms of the number of AHA! experiences. There is no conclusive explanation for this, only a hypothesis. I have used this pentominoe question a number of times before, and have read many student journals on it, and know that it is a good problem. It seems to haunt the students, simultaneously frustrating them and enticing them to keep working. I discuss this further in the section *What Makes a Good Problem*. Furthermore, the anecdotal evidence from the past, as well as within this study indicates that, without the use of algebraic symbolization and manipulation, this problem is most commonly solved when the students stop seeing the pentominoe shapes as static objects and begin to see them as dynamic collections of five squares morphing from one pentominoe to another. This shift in 'seeing', or noticing as Mason (1989) refers to it, is at the centre of students AHA!'s. This can be seen in Leanne's comments from her problem solving journal.

> *Leanne:* *I remember the moment when we GOT IT. We were looking at how the shapes either fit into a 3 x 3 box or into a 2 x 5 box and were looking at what the remainders were for each shape. It was when had organized each shape into these two boxes that I realized that when you take a star shape which has no remainder and* ***move the top block*** *one to the right the new shape will have a remainder of one.*

Perhaps problems where 'seeing', or suddenly 'seeing', is required to solve them are more prone to AHA! experiences. Such problems are certainly good models for problems that require a gestalt psychology to solve, and as discussed in chapter two, gestalt psychology of problem solving and AHA! experiences are closely linked.

Another difference between the two groups is the number of students who claimed to not have had an AHA! experience. In the secondary group there were six students (out of 34, or 18%) whereas in the elementary group there was only one (out of 38, or 3%). This is most likely due to the differences in mathematical ability between the two groups; the secondary participants were just not as challenged by the problems. The problem solving journals reveal that, as a whole, the secondary group was much more adept at simply working through the problems in a systematic fashion, relying on their vast experiences with mathematical problem solving. As such, they were getting stuck less often, and having fewer AHA! experiences. This is also evident in the number of AHA!'s claimed per individual student. Although not asked for specifically, many of the elementary participants (17 out of 38) stated that they had had several AHA!'s during the course, and they chose only to write about their most powerful or most memorable one. This can be seen in Jessica's comments.

> *Jessica:* *I have had a few AHA!'s in this course. One was while solving the Treasure of Captain Bird puzzle. I love maps. Everywhere I go I collect maps. [..] Another AHA! came when I was working on the 7, 11, and 13 problem. The AHA! came when I saw that I should multiply 7 x 11 x 13. This revealed the number 1001 and the solution to the problem. [..] I think my biggest*

> *AHA! moments, however, come when you have been working on the problem for a long time. The problem becomes so ingrained in your mind that you carry it with you wherever you go and regardless of what you are doing. Most of my AHA! moments have come to me while running. As you establish a rhythm and pace, your mind relaxes and thinking flows more freely. The key to running is that it allows your mind to relax. This is how I found the key to the pentominoe puzzle, after a run sitting on a rock at the Maritime Museum and relaxing. This was my most powerful AHA! …*

This was rarely the case for the secondary participants. Of the 34 students, only two mentioned more than one AHA! experience in the course. The rest only spoke of one, if they spoke of any.

In analysing the reflective journal entries regarding AHA! experiences at a deeper level the same themes emerged as had in the study discussed in chapter five. The particulars of the problem and the details of the actual idea that came took a back seat to discussions of feelings and what they were doing when the AHA! experience happened. Prevalent among these comments were exultations of the new-found confidence that these experiences fostered. This can be seen in the comments of Carolyn, Jennifer, Leslie, and Sherry from the elementary group.

> *Carolyn:* *This single moment bolstered my confidence in math more than anything I had done since grade 10 (try to get your head around how amazing this is because I was in grade 10 in 1973). Perhaps it was too early in the term for you to be familiar with the personalities of the students in class but up to that point I had not volunteered an answer or uttered a single word in class. Soon afterwards I was able to bring myself to interact publicly with a little more confidence.*

Jennifer: *Coming to the answer made me feel much more confident that "Yes! I can solve math problems." It is possible! I think having this experience drove me to stick with the math problems that came later in the course when I was stuck.*

Leslie: *Other than being pretty happy with myself, I generally felt more confident in my math skills. It gave me the confidence to wade in and tackle Peter's other problems, some of which I got and some of which I didn't.*

Sherry: *In further problems in the future, I found that I looked back on this AHA! moment, and that enabled me to have the confidence in doing and learning mathematics. After all, if that insight has occurred once, chances are it will happen again on future problems.*

These sentiments can also be seen in James and Christy's comments from the secondary group.

James: *The exultation of the AHA! is the joy of success, and the reason I do math. Ultimately, I suppose, it is this success that math problems all are intended to inspire, and sufficient success has addicted all of us in this class to continue to study math.*

Christy: *Needless to say this experience was pretty encouraging; a moment of definite satisfaction. It was at this point as well where I was feeling like I wasn't a mathematical thinker at a level that was at par to the class as a whole. Many of the people in the class intimidated me with their knowledge (and their weirdness) and I often felt kinda stupid. [...] After the AHA! I walked away thinking I was capable in this class and I too could solve problems.*

These comments regarding confidence, as well as other comments regarding beliefs and attitudes about mathematics in general can be analysed in the context of affect (beliefs, attitudes, and emotions) used in chapter five. As the results of this analysis are similar to those in chapter five, and for brevity sake, I will not do this. Instead I will only summarize the conclusions. The AHA! experience has a profound and significant effect on a person's beliefs and attitudes regarding mathematics as well as their ability to do mathematics. One single occurrence of this phenomenon seems to have the ability to provide even a resistant and phobic student with a more positive affect and greater confidence in himself or herself.

Problem Solving Journals

The explicit teaching to, and expectation of, the use of the three voices in the problem solving journals proved fruitful in gaining insights into the participants' problem solving efforts. For the first time in teaching these courses I was reading journals that seemed to reflect the actual problem solving processes of the students. Emotions and ideas that had not been present in the past, were now revealed in almost everyone's writing. Furthermore, and of greater relevance, AHA! experiences were discernable within their writing. In this section I present the analysis of these problem solving journals. I begin by examining the synchronization between students' claims of AHA! experiences, as presented in their reflective journals, and evidence of AHA! experiences, as existing in their problem solving journals. This is followed by a discussion as to the different ways in which these two forms of journaling complement each other.

The problem solving journals were due at the beginning of class in week 11 of the course, and the prompt to reflect on their AHA! experiences in their reflective journals was not given until the end of class on that same day. There had been no mention of the AHA! prior to them submitting their problem solving journals. Therefore, evidence of such an experience emerging in their problem solving journals, either through some exclamation by the *participant*, some accounting by the *narrator*, or some change in reasoning by the *mathematician* would represent a sincere, and unprompted, effort to represent an AHA! experience. Although I looked for such evidence while reading the problem solving journals in general, I paid particularly close attention to those sections indicated by each student's reflective journal as being a problem in which they experienced an AHA!

Twenty-five secondary participants claimed to have had an AHA! while working on either a homework problem or the term problem. Of these, 18 had corroborating evidence of their AHA!'s in their problem solving journals. That is, the use of the three voices in their problem solving journals produced sincere enough accounts to discern AHA! experiences within their writing. Likewise, of the 36 elementary participants who claimed in their reflective journals to have had an AHA! experience, 29 had evidence of it in their problem solving journals. These results are summarized in Table 5.

Table 5: Coordination of AHA!'s Between the Problem Solving Journals and Reflective Journals

course	number of students	number claiming an AHA! in their reflective journals	number who had evidence of AHA! in their problem solving journals	% of claims that had evidence
EDUC 415	34	25	18	72%
EDUC 475	38	36	29	81%
total	72	61	47	77%

There were corroborations that told the same story in both journals. Such was the case with Stephan's journaling with respect to the *Pentominoe Problem*. He has picked up on the fact that moving a single square of a pentominoe shape either left or right one square will change both the sum and the remainder by one. In his problem solving journal the following passage appeared:

> *Stephan:* *I've got it! I'm sure the half dozen beers have helped but I think I've solved it. Its simple really and I've gotten it because of, believe it or not, golf! The explanation may be muddled but it makes perfect sense (in my head). In golf there are two values when keeping score: the number of shots actually taken and the number of shots relative to par [..] How does this apply to the Pentominoes puzzle? [..] When all the blocks are vertical their sum divided by five will always be a whole number, no matter where they are on the number grid. These vertical blocks are par (E). If you then move a block to the right one then your score changes to +1. If you move it left then it changes to –1 …*

It begins with the *participant* exclaiming "*I've got it!*". This is followed by a short account by the *narrator* as to where the idea came from, and then the *mathematician* takes over in trying to articulate how and why it works. This AHA! is corroborated by the following passage in his reflective journal:

Stephan: *The AHA! came right after I'd played a round of golf and I was watching golf on TV in the clubhouse. On the screen flashed a player's scorecard and I realized that the very notion of par was the solution to the Pentominoe puzzle.*

Other corroborations told a part of the story in each of the two journals, and it took the two of them together to get the whole story. This can be seen in Marie's journals. In her problem solving journal she reveals the mathematical understanding of the *Pentominoe Problem* she has suddenly gained. Up until the passage presented below, Marie has been working on the problem at the level of numbers, moving the shapes around and finding their sums. This way of thinking suddenly changes.

Marie: *Yes! I think I have figured it out! The solution has to do with symmetry! I discovered that the cross is always divisible by 5, and I am pretty sure it is because it is symmetrical. Shoot! This doesn't necessarily work because there are other shapes like "Tee" and "Z" that are always divisible. Why! I really think that symmetry has something to do with it! But wait … "Tee" is only divisible when it is upright …*

It is clear from this passage that something has occurred to her, even though it does not work out as nicely has she had hoped. From the exclamation of the *participant* as well as from the change in reasoning be the *mathematician* it is reasonable to assume that she has had some sudden insight, perhaps even an AHA!. However, because the voice of the *narrator* is missing we need to read about this in her reflective journal to find out exactly how the idea came about.

Marie: *The most significant AHA! moment that I had so far is during the Pentominoes puzzle. I was stuck on trying to figure out what the remainder was going to be just by looking at the numbers … I couldn't possibly* *imagine that you could memorize all of the possible combinations. I had been working on the problem all day, and struggling with it, and had finally given up trying. I went out for the evening and came home and sat in the hot tub for about half an hour. Even though I wasn't consciously thinking about the problem I think that the ideas were still in my head. I honestly don't know why the idea came to me … perhaps it was because I was so relaxed and tired and not consciously struggling with the problem, but all of a sudden, I had the feeling that it wasn't about the numbers but rather about the specific configuration of the shapes [..] obviously this discovery made me feel good because this idea eventually led me to the solution.*

Between the two courses there were 14 students (seven in each course) for whom there was no corroboration. That is, they made claims about AHA! experiences within their reflective journals, but there existed no evidence of such experiences within their problem solving journals. This is not to say that these students lied in their reflective journals about having AHA! experiences. Rather, for ten of these 14 students it could be attributed to very poor journaling; they simply did not manage to assimilate the method of journaling that had been taught. Their journals were often nothing more than a presentation of the solution in the singular voice of the *mathematician*. Thus, their journals revealed nothing of how they came to their ideas and/or their solutions. The remaining four students had surprisingly good journaling abilities, and as such should have had corroboration between their two journals. This was not the case, however. For these four it may very well be the case that they were being less than truthful, either in their reflective journals or in their problem solving journals. I had anticipated that this would happen with some students, and I was actually surprised at the small number for which it did (only 4 students out of 72, or 6%).

The portions of the problem solving journals not indicated by the participants' reflective journals as containing any AHA! experiences were also read. These passages also contained evidence of AHA! experiences within them. However, without the reflective journals to confirm these findings this evidence was often inconclusive

Group AHA! Experiences

One of the things that this form of journaling allowed me to see was what I have come to refer to as a *group AHA!'s*, of which there are two types. The first type is where one member of the group has an AHA! experience while working with his or her group. The second type of group AHA! is where more than one member of the group have an AHA! experience at the same time. In this study there were four cases of group AHA!'s, all within the *elementary* group, and all while working on the *Pentominoe Problem*. In what follows I summarize each of these group AHA!'s and discuss how they are evidenced in the respective journals of the group members.

While working with Paul and Jennifer on the *Pentominoe Problem*, Leslie suddenly saw how manipulating the shapes could provide the remainders. She immediately shared this insight with her group members. While this event is featured in each of their problem solving journals, only Leslie's journal shows any evidence of an AHA! experience. The others include it as a matter of how the group came to the solution. Likewise, in their reflective journals, neither Jennifer nor Paul mention this in their reflections on AHA! experiences. Clearly, the AHA! that Leslie experienced did not transfer to the other members of the group.

For Carolyn and Gary, the story was almost identical. They were working on the *Pentominoe Problem* when Carolyn had an AHA!. She quickly explained this to Gary, who immediately worked out the mathematical reason why it works. Again, this experience is featured prominently in Carolyn's journals while not at all in Gary's reflective journal, and only in passing in his problem solving journal.

For Leanne and Jackie it was different, however. From their journals it is clear that they shared in the AHA! experience. They had been working together on the problem for some time when suddenly they saw that every pentominoe shape could be derived either "*from the vertical line or from the cross by moving just a few blocks*". They seemed to come to this insight together and simultaneously, and which ideas belong to who becames blurred.

> *Leanne:* *I remember writing very fast – almost as if I was trying to keep up with our thoughts that were flowing so quickly.*
>
> *Jackie:* *We had it! It was all about the manipulation of shapes! It was euphoric! Our discovery happened so fast once we took a different approach to our thinking! The back and forth between Leanne and myself built to such a pace that we were having a conversation but it felt as if we were truly working as one. This is unlike ANY group experience I have ever had. I must admit that my view of the synergy possible in group thinking has changed!*

Shortly after this AHA! they came to me to share their new insight. From my own recollections of this encounter, as well as from the notes I wrote to myself immediately afterwards, it was clear that they were working as one. As they were showing me what they could do they were somewhat incoherent, yet completely in tune with each other. They could finish each other's sentences and it took both of them to get the words out.

The final group AHA! is unique among these occurrences. To begin with, I witnessed it. It took place during class time while the students were working on an in-class problem. It is also unique in that two of the members recall it as being an AHA! experience while the third does not.

The groups had been playing Nim (described above) with a set of five unit cubes. Normally I start this activity with seven cubes, but because I was running a sort of teaching experiment (which I describe in greater detail in a subsequent section) I decided to start with five this time. What happened next is best described in Alicia's problem solving journal.

> *Alicia:* *We had sort of had enough of the game, so Betty suggested that we use the blocks to work on the pentominoes. She started making the shapes of the pentominoes with the cubes and she would move one block to make different pentominoe. As Betty moved the blocks Jen and I started calling out the remainders. At first we were just calling them out of memory, yet as we did this and Betty moved blocks we started to see a pattern or a system.*

I remember the three students becoming very excited and animated at this point. Because I had been anticipating that such a thing may happen, I was alert to it, and I was certain I knew what it was about. I watched from a distance and only later did I approach and ask what had happened. At this point they shared with me their newfound solution to the *Pentominoe Problem*.

Their problem solving journals regarding this event are all very consistent with what I witnessed. They each speak about how they came up with the solution and it is clear that they all came to have the same thoughts and the same ideas simultaneously. However, in reading these journals something else becomes clear. While Alicia and Jen speak of this moment with language that indicates that they saw it as an AHA! experience, Betty does not. These views are corroborated in each of their reflective journals. Yet, from what I witnessed, as well as what I heard in our conversation together, it seemed like all three had the same experience. This was not like the case with Carolyn and Leslie discussed above where they alone had the idea and they then had to share these ideas with their respective partners. These three girls did not have to explain anything to each other. For all intents and purposes, they had the same cognitive experience; they just did not all have an AHA! experiences. This is an important point that will be discussed further in the conclusions to this chapter.

Invoking the AHA!: An Activity-Based Experiment

As mentioned, there were two teaching experiments performed in the context of this study, one in each course. In this section I present the first of these two teaching experiments and discuss the results that it produced.

In the early part of chapter four I discussed my failed attempts to invoke AHA! experiences through the use of two similar problems worked on sequentially in a clinical interview setting. I now know that the clinical interview setting, itself, is not conducive to the AHA! experience in that it does not provide enough time to incubate, enough things to become distracted by, or other people to discuss the problems with. Given that I now had this greater understanding of what was necessary for the occasioning of AHA! experiences I decided to revisit the idea of using a set of similar problems, but this time in a different setting. I decided to see if I could invoke the AHA! experience in my secondary participants through the presentation of, and subsequent engagement in, two related problems worked on in a group setting with lots of time to think and many opportunities to interact. I begin by presenting these problems and discussing their solutions.

> Chessboard and Dominoes
> *A chessboard has two squares removed, one each from two opposite corners. Assuming that a domino placed on the chessboard covers exactly two squares, can the modified chessboard be tiled with dominoes?*

This is a classic problem, the solution to which requires that the solver attend to the salient, and not obviously relevant, feature of the colours of the individual squares. If this is done then it becomes clear that every domino will cover one black and one white square. It also becomes clear that the two squares that have been removed are both the same colour. There is an imbalance of black and white squares left to be covered, and thus there exists no configuration of dominoes that will cover the modified chessboard.

> Student Moves
> *The desks of 25 students are arranged in a 5 x 5 array. The teacher comes in and tells the students that everyone has to change to a different desk. The stipulation is, however, that they are only allowed to move into an adjacent seat (not kiddie-corner) and everyone must move. Can it be done?*

This is not as well-known a problem as the previous one. Although it quickly becomes clear that there may not exist a way to adhere to the conditions of the problem and still have every student move, the proof for this remains elusive. In fact, it often remains elusive until one comes up with the idea of superimposing the problem onto a chessboard. Then one realizes that every student move consists of either moving from a black square to a white square or vice versa. That is, every student on a white square must move to a black square, and every student on a black square must move to a white square. Again, however, there is an imbalance between the number of black and white squares, thereby making it impossible to do.

Although the mathematical structures of the solutions to the two problems are virtually identical, the mathematical structures of the problems are completely different. However, the two problems do have the surface feature of a square grid in common, even if it is only implicit. As such, these problems can be referred to as "*almost isomorphic*" (English, 1998, 1997; Novick, 1995, 1990, 1988; Novick & Holyoak, 1991). Therefore, it could be argued that knowing the solution to one problem would allow a person to reason their way to the solution of the other problem. When this happens the reasoning is referred to as analogical reasoning (English, 1998, 1997; Novick, 1995, 1990, 1988; Novick & Holyoak, 1991). Although such a framework for analysis exists it goes beyond the scope required to analyse the results of this teaching experiment.

In performing the experiment the *Student Moves* problem was given first and the *Chessboard and Dominoes* problem was given second. I began by asking for four volunteers to act as observers of the activity. Their role was to watch, listen, and record the efforts of their peers as they worked at solving these two problems within their groups. Once this was explained the class was given the *Student Moves* problem to work on. After 20 minutes there was only one group that had solved the problem. The observers were asked to report what they had witnessed, which they did. They described seeing their peers working to solve this problem pictorially. At first they tried to create a closed path of moves that would allow each student to move. When this failed they tried to prove that the reason it fails is because it cannot be done. Some had begun to see that the moves do not have to be in a closed path at all, that maybe an exchange system, or a series of smaller closed paths would work.

At this point the students were told to set this problem aside and to consider a new problem. I then presented them with the *Chessboard and Domino* problem. Surprisingly, only a few recalled seeing this problem in the past, and no one remembered how to do it. With heavy hinting with regards to the fact that the problem specified a chessboard and not simply an 8 x 8 grid the students began to see that perhaps the colour of the squares were important. Eventually everyone in the class arrived at a solution to the problem.

As the class was asked to revisit the first problem I called upon the observers to pay very close attention to what was about to happen. In a matter of seconds students began having AHA!'s. It was obvious from the body language and their exclamations that this was happening. Because I did not want to taint their journals I did not engage them in an explicit discussion of the AHA!

experience. Instead, I asked the observers to explain how people actually solved the *Student Moves* problem. There was no need for them to state that for many of the students the solutions had come quickly, even instantaneously. Instead, they focused on the fact that from what they could tell, no one's solution came from the direction of initial attack on the problem.

As clear as it was that the use of similar problems in the group setting was causing these students to have AHA!'s, it became equally clear that these were not very powerful experiences. This is because, of the 30 or so students present that day, only one student mentioned it in her reflective journal. Ironically, from her journal it seems that the solution came to her, not from superimposing the chessboard onto the problem, but from a totally unrelated direction.

> *Alicia:* *It wasn't until we got to come back to the question that it came to me. I was thinking "So what the hell is really going on here, anyway?" And that's when it hit me with the students and the desks: "Well, every desk will have to see two students, of course." [..] I didn't have the solution delineated in my head, but somehow I knew that my thought was one of the important things I had needed to realize. I still couldn't put the idea into words, and in fact I thought there were other things to explore to explain it properly, but at least now I was looking in the right direction and I knew I would get it.*

Alicia's very powerful AHA! had nothing to do with transferring information and strategy between problems.

Invoking the AHA!: An Extended Experiment

In the previous section it was mentioned that I engaged in two separate teaching experiments. In this section I present the second of these experiments.

One of the goals in first restructuring, and subsequently teaching, Education 415 and Education 475 had been to create an environment in which there were lots and lots of surrounding ideas; ideas that were in the air, but not necessarily anchored to each other. I decided I was going to try to do this with the elementary participants in the context of their *Pentominoe Problem*.

The first way of introducing relevant ideas into the air was through a repeating pattern activity. It begins by having all the students sit in a circle and number themselves off using consecutive numbers clockwise around the circle. The first student then goes to the centre of the circle and chooses a coloured cube from a box containing many cubes of different colours. The second student then chooses a different colour, and so on, until I decide to stop the group. At this point the next student is asked to start the colour pattern over again by choosing the same coloured block as the first student, and so on. We then engage in a discussion wherein the students are invited to formulate arguments for how to figure out what colour block a specific student will get given that they know what that student's number is. This has always been a successful activity and very nicely demonstrates the connection between counting up from a multiple and division with remainder, two strategies that research shows are not necessarily seen as connected in the minds of the students (Zazkis & Liljedahl, 2002). It also shows a connection between, for example, remainder three and one less than a multiple of four.

In doing the activity this time, I decided to focus on a repeating pattern of length five. This was chosen because the *Pentominoe Problem* required the students to consider remainders in division by five. No attention was drawn to this connection; I simply let the discussion proceed as usual. Although there was no evidence of any connections happening as we sat there in the circle, I found out later that at least one had been made. Carmel revealed in her reflective journal that she had an AHA! experience immediately after this activity.

> *Carmel:* *Fortunately, I stayed behind with my pentominoes group to work on the puzzle. We had just finished a lesson that involved a pattern activity. It used coloured Unifix cubes to create a repeating pattern around the class circle. This lesson reinforced the basics of multiplying and dividing and showed how a remainder of 3 was the same as a remainder of –2. As we began looking at our shapes I suddenly saw what was happening to the remainder of the shapes as we flipped them over. We already knew that a shape with a remainder of 1 became a remainder of 4 when we flipped it over, a remainder of 2 became a 3, and so on. But now I saw why. A remainder of 1 became a remainder of –1 when it got flipped, and –1 was the same as remainder 4 …*

Clearly, putting these ideas 'in the air' had an effect on her when she later sat down to work on the problem.

Another way in which I attempted to create surrounding information took place two weeks later. As mentioned, students do not solve the *Pentominoe Problem* until they can to see the shapes as special cases of a dynamic process with individual squares moving about to form new shapes. From observing the class I knew that not many groups had come to this idea yet. Most of the groups were still using acetate cut-outs of the individual pentominoe shapes and moving these around on the hundreds chart. So, when it came time to play Nim I chose to use cubes to play the game as a way to introduce them to a new manipulative for exploring the pentominoe shapes. Again, this was not made explicit. The results have already been described in the *Group AHA!'s* section and so they will not be repeated. Although the AHA!'s produced by this implicit method of filling the air with relevant, but unanchored, information produced far fewer AHA! experiences than the more direct method discussed in the previous section, the few that it did produce seemed to be much more powerful. This is most likely due to the fact that they had been working on the *Pentominoe Problem* for a long time (several weeks) in comparison to the *Student Moves* problem (20 minutes). As mentioned in chapter one, this increased investment of time serves to heighten the tension surrounding the unresolved effort at solving a problem, and subsequently increases the intensity of the emotions at its eventual resolution.

What Makes a Good Problem

What makes a good problem is subjective. Different people will characterize what they feel are good problems, differently. Schoenfeld (1982) has come up with a list of characteristics of a good problem, summarized below:

- The problem needs to be accessible. That is, it is easily understood, and does not require specific knowledge to get into.
- The problem can be approached from a number of different ways.

- The problem should serve as an introduction to important mathematical ideas.
- The problem should serve as a starting point for rich mathematical exploration and lead to more good problems.

Perkins (2000) feels that what is important is that a problem be 'problematic'. That is, the problem must cause the solver to get stuck. For the purposes of occasioning AHA! experiences and engaging the participants in the PROCESS of problem solving the first two of Schoendfeld's points as well as Perkins' requirement have the greatest relevance.

In analysing the participants' reflective journal responses to the prompt regarding what makes a good problem, as well as what were their favourite problems several themes emerged. The first such theme is that the elementary participants liked the *Pentominoe Problem*. In fact, it was picked as the favourite problem by 30 of the 38 students (the 20 that had an AHA! experience while solving it, as well as 10 students who had AHA! experiences in other problems). As much as these students liked the *Pentominoe Problem* the secondary participants disliked the *Bit String* problems. All but a handful of students explicitly mentioned this problem as being the one that they disliked the most. In looking at the characteristics of these two problems they appear to be quite similar. They are both accessible, they have multiple approaches, they lead to more questions, and they both cause the students to get stuck. However, there is one characteristic that is quite different. In the *Pentominoe Problem* the students very quickly move to a level of understanding that regardless of where a pentominoe shape is placed on the hundreds chart its remainder will remain constant provided the orientation remains the same[26]. This allows the students to move quickly beyond the level of calculation and onto the level of conjecturing. The *Bit String* problems resist this sort of move. The students spend a great deal of time performing very mechanical calculations in order to build up enough information to even begin to look for patterns. Furthermore, they remain in this calculation mode almost right up to the point when they solve the problem. As such, conjecturing is difficult to check.

A more obvious difference between the two problems is the number of students who resolved the problems to a level satisfactory to themselves. All of the elementary participants solved the *Pentominoe Problem*, and all of them were satisfied with their solutions. The same cannot be said for the secondary participants' success rate on the *Bit String* problems. Several students did not solve the problems at all, and of those that did, very few felt satisfied with their solutions. This is especially true of the first *Bit String* problem, the solution to which is more of an algorithm for reducing the number of calculations than a closed solution.

Another theme that emerged has to do with problems that produced a love-hate relationship among the students. For the elementary participants this was the *Cats and Rats* problem. This problem confounded the students and, as such, it became the most memorable of all problems for them. Long after it was due I was receiving emails from students wanting to know exactly what the

[26] This is a direct consequence of the number of columns (10) being a multiple of the number of items to be summed (5).

correct answer was. I even received an email from the husband of one of my students explaining that his family was divided over this problem and they needed an answer to resolve the conflict. These feelings are best captured in Karleen's comments.

> *Karleen:* *I think I had a love hate relationship with the rat and cat problem. I discussed this problem with whoever would listen to me. My sister made up a rule that I was no longer allowed to bring up math at the dinner table. [..] The most memorable problem* ***by far*** *was the cat and rat problem, so many people had different answers and they were (including myself) so passionate about their answers, their process and they all made sense too. I had my Dad, my boyfriend and people at work all trying to figure out this problem. We all got different answers!*

For the secondary participants, the problem that invoked these same sorts of feelings of love-hate was the *Giant Wheel* problem. A characterization of how the participants engaged in the problem follows. Upon being given the problem everyone immediately has an answer. They then change this answer three times in the first 15 minutes of thinking about it. Once this passed the students settled on the answer they liked and then found mathematical arguments to defend their position. However, they were never so settled on their answer that they could not be persuaded by someone else's arguments, and so the arguments persisted. This can be seen in Carl's comments.

> *Carl:* *I think the most memorable problem was the Rolling Wheel problem because I spent a lot of time going in the wrong direction and a lot of time trying to convince everyone else that it was the right direction. I look back and shake my head at how much mental anguish this problem caused, for me and my classmates.*

The final theme that I discuss is what the students, themselves, said makes a good problem. In general they spoke of the first two characteristics on Schoenfeld's list; the problems should be accessible and approachable from a number of different directions. However, they also mentioned something that was not on Schoenfeld's list; *talking*. Of the 67 students who responded to this question 62 of them expressed, in one way or another, that they liked a problem that causes them to discuss things. This can be seen in Debbie's and Shirley's comments on what makes a good problem in general.

> *Debbie:* *Students should feel compelled to talk about it. The dialogue that then arises makes the learning experience much richer.*

> *Shirley:* *Discussion is an important part of learning and understanding math; therefore, a good problem will encourage and require discussion in order to solve it.*

It can also be seen in Betty's discussion of why she liked the *Cats and Rats* problem.

> *Betty:* *There was an awful lot of dialogue generated from this question. We all challenged each other's answers and tried to justify our way of doing it.*

The Problem Solving Process

Reflecting on one's own thinking processes can be difficult. Nonetheless, when asked to reflect on their problem solving processes the participants responded well. Their reflections were very good; they were insightful and detailed, and most importantly, they were honest. Again, themes emerged from their responses.

As discussed in chapter two, there is no such thing as a definitive and comprehensive problem solving process. Problem solving is a delicate balance between following obvious and well-trodden paths, attention to subtleties and non-salient details, semi-random explorations, chance encounters, and unexplained moments of insight and illumination. These sentiments are echoed in the journals of many of the students, as can be seen in James' entry.

> *James:* *My problem solving process is really anything but systematic. When presented with a problem I first scan through it quickly to check for familiar elements. If the problem is not immediately obvious, I read it through again, somewhat more sensitive to the language of it. Most of the time, by this second reading, I can link the problem to something in my experience … be it a previous problem, a technique discussed in class, or an example done in the text. If I have connected this problem to such an experience, its solution is easy. If not, I'm a touch embarrassed to say I cast about like a dog searching for a scent. I latch on to the first idea I have and worry it into formalism for a while, casting about for inspiration. On rare occasions, the move to formalism solves the problem without me noticing, but usually I linger in a purgatory of uncertainty until:*
> (a) *The light of intuition guides me somewhere.*
> (b) *I get frustrated and ask for help.*
> (c) *I leave the problem for a time to work on something else.*

Melissa echoes these sentiments, but in a slightly less refined way.

> *Melisa:* *I usually stand back and formulate the best approach based on what is known in the problem and what skills I posses. This has always worked for me in the past. In elementary and secondary school, problems were always closely connected to skills you had just*

learned, so this strategy worked well. It also worked well for some of the problems in this course, like the Fibonacci Sequence, the Mother/Daughter Tea and the Treasure of Captain Bird. It didn't work with the Math Magic problem, the card tricks, and a whole bunch of others. I now call these "charge in and try" problems. There is no strategy, you just have to get in there and do it.

Over all, there were 43 students between the two classes who, in one way or another, rejected the idea that there existed a systematic way in which to solve problems. For some of these students, like Melissa, this was a new idea that stemmed from their encounters with the non-curriculum driven problems I used in my courses. For others, like James, these ideas seemed to be more established. Either way, however, for these students there was a clear understanding that problem solving was an amorphous thing that changed with every situation.

The other theme that emerged regarding the students' problem solving processes has, again, to do with talking. Almost all of the elementary participants who responded to this prompt (31 out of 34 respondents) mentioned that talking to someone was a key part of their problem solving process. This is nicely captured in Helen's response.

Helen: *If I couldn't see anything, I'd begin talking with people. Through sharing with others I would receive new ideas which would enable me to make connections to solve the problem. Most problems I didn't solve directly on my own; I would work on my own and then discuss with another person where I was stuck. I'd usually have a brainwave when I'd be interacting with others.*

Helen, like many of her peers, mentions that this is something she does after she gets stuck. Others, like Jennifer, use talking even in the early stages of the problem solving process.

> *Jennifer:* *Lastly, to solve problems, I like to talk them over with another person – even if the other person is only listening and not talking at all. I find this extremely helpful, for it helps to clarify and focus my thoughts. Talking it out loud really helps me to see patterns I had previously been "blind to".*

On the other hand, almost none of the secondary participants (5 out of 32 respondents) mentioned talking as a part of their strategies. In hindsight, this is not surprising. My recollection of the secondary participants is that they were much more solitary in their efforts to solve problems, preferring to work on their own. They shared their successes more easily than their failures. As such, talking was a means of posturing rather than a means of problem solving.

Conclusions

From the very begin of my research into the AHA! experience I have struggled with the issue of capturing the phenomenon in some accurate fashion. There have been many failed attempts, from audio recorded clinical interviews to video recorded group work. Finally, I feel that I have succeeded.

The use of problem solving journals that incorporate the voices of the three personas – *the narrator, the mathematician,* and *the participant* – in conjunction with reflective journals has been effective in identifying the occurrence of AHA! experiences within the problem solving processes of the participants in this study. Of the 61 participants who claimed to have had an AHA! experience in their reflective journals, 47 of them (77%) displayed evidence of this experience in their problem solving journals. This evidence appeared as exclamation by *the participant*, some accounting by *the narrator*, some change in reasoning by *the mathematician*, or some combination of the three.

From this very rich anecdotal data the results of the study presented in chapter five regarding the impact on the participants' affective domains was reconfirmed. In addition, the methodology applied to the structuring of the course with respect to problem solving proved to have an impact on the occasioning of AHA! experience. The use of ongoing and frequent peer interaction, avoidance of details, focus on transmission of information through talking, and the careful selection of engaging and 'problematic' problems led to a large number of reported occurrences (61 in total).

Furthermore, the anecdotal responses to prompts produced rich discussion on the processes of problem solving, as perceived by the participants, as well as on what qualities make a good problem. This discussion clearly showed that the participants view peer interaction as being central to the process of problem solving. This characteristic was also identified as being central to the criteria for assessing the quality of a problem. In particular, it was determined that a 'good' problem is a problem that incites discussion.

CHAPTER 8

CONCLUSIONS AND CONTRIBUTIONS

Although each of the studies presented in chapters five through chapter seven has its own related conclusions, these conclusions also speak to each other and serve to influence the analysis of the studies as a whole. The three studies, their independent conclusions, and the whole of the knowledge gained serve to produce some conclusions regarding the research questions presented in chapter three. Because of the inextricable link between these questions, however, I have chosen to not present these conclusions as answers to the individual research questions per se. Rather, I present them in the context of contributions. The literature and the research presented in the preceding pages combine to create three such contributions: a theoretical contribution, a methodological contribution, and a pedagogical contribution. Within these three contributions each of the research questions is answered.

Theoretical Contribution

My main assertion is that the essence of the AHA! experience in mathematics in general, and in problem solving in particular, is in the affective domain. That is, what sets the AHA! experience apart from other mathematical experiences is the affective component of the experience, and ONLY the affective component. This is counter to the general understandings of the nature of the AHA! experience. In the literature, AHA! experiences, along with thair cadre of extra-logical processes, have always been dealt with in the context of the cognitive domain. That is, it has always been assumed that what makes these experiences extraordinary is the hidden cognitive processes that produce these extraordinary ideas. I assert that this is not the case. What serves to make the experiences extraordinary is the affective response invoked by the experience of an untimely and unanticipated presentation of an idea or solution, not the mystery of the process, and not the idea itself. I support this assertion along several fronts, culled from both the data and the literature presented in the preceding chapters.

Before I begin this discussion, however, it needs to be acknowledged that, clearly, there is a cognitive component to the AHA! experience. After all, it is the arrival of an idea that punctuates the phenomenon. The argument that follows does not deny this. What the argument does do, however, is to assert that, while the cognitive component of the AHA! experience is inconsequential to the differentiation of the AHA! from other cognitive experiences, the affective dimension is not. I begin by establishing the role that the affective dimension plays in this distinction. This is then followed by a lengthier discussion detailing how the cognitive dimension plays a relatively little role.

In chapter five it was shown that the emotive response created by AHA! experiences is capable of transforming the resistant students' beliefs and attitudes towards mathematics as well as their beliefs and attitudes about doing mathematics. These findings were reconfirmed through the study presented in chapter seven. The mathematicians were also not hesitant to discuss the feelings brought about by their experience. In their case they did not have negative beliefs and attitudes to overcome. This did not, however, prevent them from having a positive response to their experiences. They too spoke of excitement, joy, and satisfaction in their accounts of insight, illumination, and intuition. Combined, these studies answer the research question regarding the effect that an AHA! experience has on a learner. Clearly, it has a positive effect on students and mathematicians alike. In addition, if the learner has negative beliefs and attitudes, it could have a transformative effect.

This transformative effect, alone, makes the affective response invoked by the AHA! experience very different from the responses invoked by other mathematical experiences. The literature on affect and mathematical experiences indicates that success and feelings of accomplishment contribute to a change in beliefs and attitudes. However, it is suggested that the change they produce is minute, and long periods of sustained and successive success are required to create significant change (Eynde, De Corte, & Verschaffel, 2001). The study presented in chapter five shows that AHA! experiences can produce changes in beliefs and attitudes very quickly, in the time it takes for an insight to be verified. This indicates that the affective responses to a single AHA! experience is much more powerful than the affective responses to an instance of success in mathematics (not consisting of an AHA!).

Having established the contribution of the affective dimension to the distinction of the AHA! experience, I now argue the cognitive dimension's role is relatively insignificant. The most obvious evidence for this comes from the fact that nowhere in the data were the details of the idea central to the presentation of the phenomenon. This was true for both the students and the mathematicians. In the few situations that the details were presented, their contribution to the account of the experience was inconsequential. This is consistent with the words of Poincaré (1952) wherein he clearly states that the details are unimportant.

> *I must apologize, for I am going to introduce some technical expressions, but they need not alarm the reader, for he has no need to understand them. I shall say, for instance, that I found the demonstration of such and such a theorem under such and such circumstances; the theorem will have a barbarous name that many will not know, but that is of no importance. What is interesting for the psychologist is not the theorem but the circumstances* (p. 52).

As such, the cognitive products of the AHA! experience, although clearly not absent, do not contribute to description of the experience. This is further supported by the existence of false AHA!'s, a phenomenon discussed in chapters one and five. False AHA!'s are occurrences of the AHA! experience wherein the products of the experience (the ideas) turn out to be incorrect. However, neither the occurrence or the intensity of the AHA! is diminished by this lack of correctness. That is, the idea is irrelevant to the phenomenon.

In fact, the only time that ideas or solutions are mentioned at all by the mathematicians is in the context of describing the AHA! experience as the sudden appearance of an idea. These descriptions can be sorted into two cases. The first case speaks of ideas in the context of how and when they come to mind as expressed by Dan J. Kleitman.

> *And relevant ideas do pop up in your mind when you are taking a shower, and can pop up as well even when you are sleeping.*
>
> \- Dan J. Kleitman

The second case speaks of the significance or importance of the ideas that come to mind. This is demonstrated by Wendell Fleming's comment.

> *What I think you mean by an AHA! experience comes at the moment when something mathematically significant falls into place.*
>
> \- Wendell Fleming

The first of these cases is no different than the situation described in the previous paragraph. The second case, on the other hand, seems to place the nature of the idea in a much more central role. However, as was discussed in the concluding remarks of chapter six, significance in the face of an AHA! experience is not a measure of the quality of an idea but rather a 'sense' that is invoked by the way in which the idea comes to mind. In chapter six I referred to this as a 'sense of significance' and compared it to the 'sense of certainty' that also accompanies the phenomenon. As such, even the significance of the idea is an affective response to the AHA! experience.

Further evidence of the insignificance of the idea to the contribution of the phenomenon of the AHA! can be found in the discussion of group AHA!'s presented in chapter seven. In particular, I am referring to the group AHA!'s where all three members of the group had the same idea, at the same time, and in the same flash of insight. That is, to all intents and purposes the three members of the group had the same cognitive experience. In the end, however, only two of the group members, Alicia and Jen, viewed the event as an AHA! experience. The third member, Betty, did not. She identified the moment of insight in her journal, but saw it simply as another idea coming to mind. Furthermore, in comparing Betty's description of the event with that of her partners the only distinction between the accounts lay in the affective descriptions of the experience. Alicia and Jen, who claimed it as an AHA! experience gave accounts rich in affective descriptions, while Betty's descriptions were completely devoid of any affective elements whatsoever. In the concluding remarks of chapter seven I argued that the reason that this occurred is that although the three students all had the same idea come to them at the same time, and in the same way, their affective response to that idea varied. This is not dissimilar from people's varying responses to events in general. Any time a group of people share in an experience – whether it be a movie, a concert, a play, an accident, or *an idea* – they will all have different affective responses to that expereince. Some will like it, some will not, some will be indifferent, and so on.

Finally, I offer the very succinct, and definitive, comment from Henry McKean.

> *No, I don't find it different from understanding other things in life!*
>
> \- Henry McKean

McKean's view is that the understandings gained from AHA! experiences are no different than the understandings gained from other sources. This is not only in keeping with the arguments posed above, but also brings forth the question of the origins of ideas in general. As presented in chapter one, Hadamard (1945) argued this very question in terms of the antechamber of the mind. He posed that, if pushed back far enough, the origins of every idea, even of every spoken word, is the product of the mysterious and wondrous workings of the mind. As such, at the level of origin the ideas produced by AHA! experiences are no more extraordinary than the origins of any other idea, or even of this sentence.

Together all this evidence speaks to the very *essence* of the AHA! experience. An AHA! experience is an affective response to a cognitive event, and like any other affective response, it differs in intensity depending on the individual as well as the situation. What makes it special, is not the idea itself, but rather the way in which the idea comes to us, with "characteristics of brevity, suddenness, and immediate certainty" (Poincaré, 1952, p.54).

Methodological Contribution

From the results of the study presented in chapter seven it is clear that the method of journaling that I developed to track participants' mathematical problem solving processes is an effective data collection instrument. As such, it is a methodological contribution to the mathematics education research community at large.

Mathematical problem solving in general, and the AHA! experience in particular are difficult phenomena to track. Methodologies such as think-aloud protocols (Weber, 2001) and audio and video capture are effective strategies for capturing intentional and conscious problem solving processes in the clinical interview setting. However, as discussed in chapter two, only a small part of problem solving is conscious and intentional. There are also periods of incubation and moments of illumination to contend with, neither of which flourishes in the clinical interview setting. Incubation takes place during times of non-mathematical activity and illumination occurs at untimely moments of daily life such as taking a shower or driving. Thus, true problem solving requires that the participants be given lots of *time* and *space* to engage, rest, and reengage with the problem. This alone makes direct observation of the problem solving process problematic. A tracking method that allows for the accurate capture of the problem solving process while at the same time not restricting the participant's sincere engagement in the problem is needed for the collection of data.

The use of the three personas – *the narrator, the mathematician,* and *the participant* – that I developed for the collection of data regarding the AHA! experiences is such a tracking mechanism. By being able to write using the voices of any combinations of these three personas, participants are able to record their efforts in both an accurate and a truthful manner. Furthermore, the journal provides data that is self-triangulating. Changes in thinking or strategies will reveal themselves in each of the three personas. Done correctly, *the narrator* will explicitly announce a new direction of thought. Meanwhile, *the mathematician* will demonstrate this new direction in the form of a schism in either the logic of his or her argument or a change in direction in his or her presentation of mathematics. Finally, *the participant* will make some form of emotive exclamation signalling the presentation of a new idea.

The value of this form of journaling as a data collection instrument can be further enhanced through the corroboration of a reflective journal. Such a journal, used to describe mathematical episodes in the form of retroactive introspections, will provide a further degree of validity to the accounts presented through the use of the three personas. Together the use of these two forms of journaling provide a quality of data regarding participants' efforts and thought processes that is valuable to researchers interested in problem solving, in general, and the extra-logical processes, in particular.

Pedagogical Contribution

As presented in chapter five and subsequently confirmed in chapter seven, the AHA! experience has a powerful transformative effect on the negative beliefs and attitudes of resistant students, both with regards to the subject of mathematics and their ability to do mathematics. Furthermore, research shows that student achievement in mathematics is strongly linked to both of these aspects of the affective dimension (Leder, 1992; Ponte, Matos, Guimarães, Cunha Leal, & Canavarro, 1992). Hence, the incorporation of this phenomenon into the ongoing culture of the mathematics classroom experience would be a valuable strategy for improving both student achievement and student enjoyment in mathematics.

From the data presented in chapter six I extracted, among other things, what the mathematicians indicated were characteristics that heighten the chance of an AHA! occurring. The effectiveness of these characteristics to create a problem solving environment conducive to the occasioning of AHA! experiences was then studied, the details of which are presented in the chapter seven. The findings of this study indicate that that AHA! experiences occurred through facilitation of ongoing and frequent peer interaction, avoidance of details, focus on transmission of information through talking, and the careful selection of engaging and 'problematic' problems. Furthermore, these experiences occurred in large numbers and were powerful enough to trigger the changes in affect mentioned above.

Although it is not possible to guarantee that any one approach, or any one problem, is going to trigger an AHA! experience for any one student, the overall result indicates that using the instructional approach summarized above and detailed in chapter seven do support the occurrence of AHA! experiences on the whole. Furthermore, there is nothing extraordinary about any of the practices required for this instructional approach. This means the ability to create an environment conducive to the occasioning of AHA! experiences is available to any classroom teacher.

CHAPTER NINE

IMPLICATIONS FOR THE TEACHING AND LEARNING MATHEMATICS

Given the conclusions in the previous chapter, as well as the findings of the individual studies, I would be remiss if I did not explore the implications that these conclusions have to the teaching and learning of mathematics. Some of these implications are specific to the AHA! experience, others to the more general context of problem solving, and others yet, to the context of mathematics in general. I begin with those implications that are most general.

'Doing' Mathematics

The responses from the mathematicians very clearly indicate that 'doing' mathematics and problem solving are synonymous. Given this, it raises the question, '*what then is the K-12 curriculum filled with?*' The answer to this is artefacts. The K-12 curriculum is filled with the artefacts of 'doing' mathematics. That is, it is filled with the facts and the skills produced from the mathematical 'doings' of people long ago and far away, and long since dead. This is not to say that some of the facts and skills are not important, but rather that they are not to be mistaken for 'doing' mathematics. For this reason alone, problem solving needs to be brought forth in the mathematics classroom as THE activity central to mathematics.

Humanizing Mathematics

Humanizing mathematics is often seen as the introduction and presentation of human involvement in the development of mathematical knowledge, either through curiosity or necessity (Egan, 1997). Most often this takes the form of a historical or cultural approaches to mathematics (Percival, 2001). From the work presented here it can be seen that the humanizing of mathematics need not be centred in other humans' involvement of mathematics and mathematical activity, but in the individual student's own involvement in mathematics. Allowing students to engage in a practice of problem solving that honours the extra-logical processes in general, and the AHA! experience in particular will allow every child to "experience at a few moments [..] the power and excitement of mathematics, [..] so that [..] he at least knows what it is like" (Wheeler, 1975). As such, they can come to appreciate the appeal that 'doing' mathematics has for humans.

Implication for Teaching to the AHA!

It is clear from my research that the AHA! experience has a powerful and transformative effect on resistant learner's beliefs and attitudes regarding mathematics. Couple this with the fact that past research has repeatedly shown that beliefs and attitudes are strongly linked to school achievement (Leder, 1992; Ponte, Matos, Guimarães, Cunha Leal, & Canavarro, 1992) and the result is a very strong argument for making explicit effort to occasion AHA! experiences. The details of how to go about this at the level of the classroom teacher were presented in chapter eight and I will not repeat them here. However, I will discuss the need to facilitate this phenomenon at the level of curriculum and educational policy.

The AHA! experience is situated in the larger context of problem solving. As such, the first step towards occasioning the AHA! phenomenon is to present students with interesting and challenging problems. Furthermore, if the AHA! experience is to become a regular occurrence in the classroom, then problem solving needs to be given a central role in mathematics curriculum and a more central role in mathematics instruction. However, this means more than simply making problem solving a unit, even if that unit is a substantial one, to be studied every year. Problem solving needs to be integrated, in every sense of the word, into the daily routines of mathematics instruction. It needs to be a theme that runs through every unit at every grade and it needs to become a significant part of what it means to do mathematics.

None of these needs regarding the central role of problem solving can be achieved unless steps are taken at the policy level. Not only does the use and role of problem solving need to be made explicit within policy documents, its use needs to be accommodated by a decrease of other demands on curriculum and teachers. This can only be achieved through a decrease in the amount of 'noise' present in existing mathematics curriculum, that is, a reduction in the cacophony of disconnected and meaningless particulars surrounding topics in mathematics.

There needs to be time and space provided for the sincere and meaningful engagement in mathematics topics through problem solving, rather than through the systematic dispensing of items on a list of details.

Implications for Teaching Problem Solving

Although I discussed the need for problem solving in the occasioning of AHA! experiences in the previous section, I did not discuss the teaching of problem solving per se. As discussed in chapter two, there are many different ideas for teaching problem solving. From the prescriptive methodology of Pólya in *How to Solve It* (1957) to the very holistic approach of Gestalt psychology (Koestler, 1964) they all have unique qualities that they offer to the discourse on problem solving. Pólya saw problem solving as a recipe, a set of steps that when taken will lead to a solution. In contrast, Gestalt psychology states that problem solving cannot be taught, cannot be distilled down to a set of rules; instead the problem must be turned over and over in the mind until a direction of attack presents itself. These strategies are the extremes on a continuum, with Pólya at one end and Gestalt at the other. Situated between these two extremes is the work of Alan Schoenfeld (1985), John Mason et al (1982), and David Perkins (2000). What they all have in common is that, with the exception of Gestalt psychology, each of these heuristics is accompanied by a teaching methodology that actualizes their descriptive frameworks. However, they differ with regards to how much, if at all, they honour the role of the AHA! in problem solving.

The heuristics of Pólya (1957) and Schoenfeld (1985) do not acknowledge the extra-logical processes of problem solving at all. These heuristics are built on the understanding that if a problem is to be solved, it will be done using prior knowledge, as well as intentional and mechanical methods. As has been demonstrated in the studies presented here, however, this is not always the case. The extra-logical process of the AHA! experience very clearly contributes to the problem solving processes of mathematicians as well as mathematics students. Thus, if a problem solving heuristic is to be taught explicitly then one needs to be chosen that works in conjunction with AHA!'s rather than instead of AHA!'s. Either the heuristics of Perkins (2000) or Mason et al (1982) is an excellent choice for this task. Both of these heuristics not only honour the contribution of the AHA! to the problem solving process, but they incorporate it into their respective methodologies.

Implications for Problem Solving Assessment

If problem solving is to become a central part of the teaching and learning of mathematics then the process in which a student engages will inevitably have to be evaluated. The problem with this is that traditional assessment tools drastically limit the amount of time that is available for a given task. Unfortunately, as was shown in the various studies presented here, limited time means limited results. True problem solving requires large amounts of time, not just for the directed efforts of intentional and conscious work, but also for the ensuing unconscious work. Such being the case, assessment instruments need to be used that do not constrict, too tightly, the amount of time available for problem solving.

Furthermore, because a large part of true problem solving utilizes both the conscious and unconscious workings of the mind, and thus relies on the extra-logical processes of mathematical thinking, there are very few guarantees that answers will be forthcoming. Hence, assessment instruments also need to place a greater value on the processes of problem solving than the products of problem solving.

I propose that one instrument that will satisfy both of these aforementioned requirements is the journaling method I developed, and presented, in chapter seven. By incorporating this form of journaling into the problem solving process, not only does the process become transparent, but also the need to limit the time allowed is dispensed with. Students can experience problem solving as it is meant to be experienced all the while recording their efforts in the voices of the three personas: *the narrator, the mathematician,* and *the participant*. In the end, their journals should tell a story of working through failed attempts and wrong turns, progressing through the problem in fits and spurts, and oscillating between periods of inactivity, stalled progress, rapid advancement, and AHA!'s. By providing a way in which their process can be captured, it can be evaluated, and thus, it can be valued.

AFTERWORD

And so the journey ends. Along the way there were many distractions, many digressions, and many pursuits of varying fruitfulness. I have come far, and strayed further. I have ended up in a place very different from where I started and even further from where I thought I would end up. In many ways this journey has been my own personal AHA! experience, punctuated by the sudden realization that the AHA! is not a phenomenon to be *defined*, but a phenomenon to be *described*. In the aftermath of this realization I have seen that the essence of the AHA! experience lies not in the mysteries of the *cognitive domain* but in the wonders of the *affective domain*. I have seen that its power lies not in what it makes a person *think*, but what it makes them *feel*. I have seen that it is not to be *controlled*, but to be *occasioned*.

This dissertation has been the story of my AHA! experience and, like the AHA! experiences of my students, it has been told in the voices of three personas. In this case, however, these personas have not been the *narrator,* the *mathematician,* and the *participant*. Instead they have been the *narrator,* the *researcher,* and the *teacher*. The *narrator* has moved the story along providing the chronology and the circumstances of the variety of studies that has made up the journey. The *researcher* has provided the details of the various studies, including the justification of methodologies, the analysis of the data, and the realization of its conclusions. The *teacher* has explored the implications of these conclusions, as well as an evolving understanding of the AHA! experience, on the teaching and learning of mathematics in the mathematics classroom.

But there is more to learn. To begin with, it was established within this dissertation that the AHA! experience has a positive and, sometimes, profoundly transformative effect on a learner's beliefs and attitudes about mathematics and their ability to do mathematics. The extent to which this effect is retained over time has yet to be determined. Likewise, the degree to which this effect is retained in the face of negative mathematical experiences has also to be explored. Some of the participants in the studies presented within this work were preservice teachers. As such, their experiences with the AHA! could effect their ensuing teaching practices. The extent to which this is true needs to be examined. And so a new journey begins.

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APPENDIX A

ETHICAL APPROVAL

APPENDIX B
CONSENT FORMS

Consent Form: Preservice Secondary Mathematics Teachers

SIMON FRASER UNIVERSITY

Informed Consent By Participants In a Research Project
"The AHA experience: mathematical contexts, educational implications"

The University and those conducting this project subscribe to the ethical conduct of research and to the protection at all times of the interests, comfort, and safety of subjects. This research is being conducted under permission of the Simon Fraser Research Ethics Board. The chief concern of the Board is for the health, safety and psychological well-being of research participants.

Should you wish to obtain information about your rights as a participant in research, or about the responsibilities of researchers, or if you have any questions, concerns or complaints about the manner in which you were treated in this study, please contact the Director, Office of Research Ethics by email at hweinber@sfu.ca or phone at 604-268-6593.

Your signature on this form will signify that you have received a document which describes the procedures, possible risks, and benefits of this research project, that you have received an adequate opportunity to consider the information in the documents describing the project or experiment, and that you voluntarily agree to participate in the project.

Any information that is obtained during this study will be kept confidential to the full extent permitted by the law. Although you will be required to write your name on the journals, your identity is not part of the data, per se, and thus will be omitted from any documents produced as a result of this study. Materials will be maintained in a secure location until they are returned to you.

Having been asked to participate in a research project or experiment, I certify that I have read the procedures specified in the information documents, describing the project or experiment. I understand the procedures to be used in this experiment and that there are no personal risks to me in taking part in the project or experiment. I understand that I may withdraw my participation at any time. I also understand that I may register any complaint with the Director of the Office of Research Ethics or the researcher named above or with Kelleen Toohey, the Director of the Graduate Studies in the Faculty of Education (8888 University Way, Simon Fraser University, Burnaby, British Columbia, V5A 1S6, Canada). I may obtain copies of the results of this study, upon its completion by contacting the investigator Peter Liljedahl. I have been informed that the research will be confidential. I understand that my supervisor or employer may require me to obtain his or her permission prior to my participation in a study of this kind. I understand that I am to allow entries from my journals and/or excerpts form clinical interviews to be used as data in a research study.

I further understand that my involvement in this research or my refusal to participate has absolutely no relation to the grading of my assignments in **EDUC 415** and to the grading of the course.

NAME: ____

ADDRESS: ____

____

SIGNATURE: ____

DATE: ______________________

Consent Form: Preservice Elementary Mathematics Teachers

SIMON FRASER UNIVERSITY

Informed Consent By Participants In a Research Project
"The AHA experience: mathematical contexts, educational implications"

The University and those conducting this project subscribe to the ethical conduct of research and to the protection at all times of the interests, comfort, and safety of subjects. This research is being conducted under permission of the Simon Fraser Research Ethics Board. The chief concern of the Board is for the health, safety and psychological well-being of research participants.

Should you wish to obtain information about your rights as a participant in research, or about the responsibilities of researchers, or if you have any questions, concerns or complaints about the manner in which you were treated in this study, please contact the Director, Office of Research Ethics by email at hweinber@sfu.ca or phone at 604-268-6593.

Your signature on this form will signify that you have received a document which describes the procedures, possible risks, and benefits of this research project, that you have received an adequate opportunity to consider the information in the documents describing the project or experiment, and that you voluntarily agree to participate in the project.

Any information that is obtained during this study will be kept confidential to the full extent permitted by the law. Although you will be required to write your name on the journals, your identity is not part of the data, per se, and thus will be omitted from any documents produced as a result of this study. Materials will be maintained in a secure location until they are returned to you.

Having been asked to participate in a research project or experiment, I certify that I have read the procedures specified in the information documents, describing the project or experiment. I understand the procedures to be used in this experiment and that there are no personal risks to me in taking part in the project or experiment. I understand that I may withdraw my participation at any time. I also understand that I may register any complaint with the Director of the Office of Research Ethics or the researcher named above or with Kelleen Toohey, the Director of the Graduate Studies in the Faculty of Education (8888 University Way, Simon Fraser University, Burnaby, British Columbia, V5A 1S6, Canada). I may obtain copies of the results of this study, upon its completion by contacting the investigator Peter Liljedahl. I have been informed that the research will be confidential. I understand that my supervisor or employer may require me to obtain his or her permission prior to my participation in a study of this kind. I understand that I am to allow entries from my journals and/or excerpts form clinical interviews to be used as data in a research study.

I further understand that my involvement in this research or my refusal to participate has absolutely no relation to the grading of my assignments in **EDUC 475** and to the grading of the course.

NAME: ____

ADDRESS: ____

____

SIGNATURE: ____

DATE: ________________________

Consent Form: Undergraduate Mathematics Students

SIMON FRASER UNIVERSITY

Informed Consent By Participants In a Research Project

"Re-learning mathematics: The reconstruction of knowledge by pre-service elementary school teachers"

The University and those conducting this project subscribe to the ethical conduct of research and to the protection at all times of the interests, comfort, and safety of subjects. This form and the information it contains are given to you for your own protection and full understanding of the procedures**.** Your signature on this form will signify that you voluntarily agree to participate in the research project.

This research in essential in order to understand how preservice teachers learn mathematics and the effect of specific instructional approaches on students' learning.

Participants in this research will give their permission to use their work in the course MATH 190 - Principles of Mathematics for Teachers - as research data. This includes observation of students' work in class and in the lab, analysis of students' journal entries, homework, projects and exams, and conducting interviews with individual students.

Materials collected in this research will be held in a secure location and will be destroyed after the completion of the study. Participants' identity will be protected in case of publication of the research results or conference presentations.

I understand that I may withdraw my participation in this research at any time.

I also understand that I may register any complaint I might have about the experiment with Dr. Rina Zazkis or with Dr. Robin Barrow, Dean of the Faculty of Education at Simon Fraser University.

I may obtain copies of the results of this study, upon its completion, by contacting:
Rina Zazkis

I understand that my identity is protected in case of publication or presentation of research results.

I agree to participate by allowing to observe my work in class and in the lab, to analyze my homework, exams and other assignments, and by being interviewed on the material related to the course. I agree that these interviewees be audiotaped/videotaped. I understand that in order to protect my identity the research data will be kept in a locked file cabinet and the pseudonyms will be used if the research leads to publication.

I further understand that my involvement in this research or my refusal to participate has absolutely no relation to the grading of my assignments in **MATH 190** and to the grading of the course.

NAME (please type or print legibly):_________________________________

ADDRESS: ____

____

SIGNATURE: ______________________

DATE: _________________________________

APPENDIX C

INTERVIEW QUESTIONS USED IN CLINICAL SETTING

The Calendar Problem:

I've chosen a calendar page, October 2000, and I'm going to place a red marker on the 1, a blue on the 2, a green on the 3, and a yellow on the 4. Now, I'm going to repeat this pattern; red on the 5, blue on the 6, green on the 7, and yellow on the 8.

a. What colour will number 13 be?
b. What colour will number 28 be?
c. If the calendar continued on forever, what colour would 61 be?
d. What colour would 178 be?
e. What colour would 799 be?
f. If there were five colours (red, blue, green, yellow, and black), what colour would 799 be?
g. If there were six colours, what colour would 799 be?

The Sequence Problem:

a. Consider the sequence 1, 5, 9, …What will the next few numbers in the sequence be?
b. Will the number 48 be in this sequence?
c. Will 63 be in the sequence?
d. Can you give me a big number that you know for sure will be in the sequence?
e. Consider the sequence 5, 12, 19, … Is 96 going to be in this sequence?
f. Can you give me a big number that you know for sure will be in the sequence?
g. Consider the sequence 8, 15, 22, … Can you give me a big number that you know for sure will be in the sequence?
h. Consider the sequence 15, 28, 41, … Is 1302 going to be in this sequence?

Comparison of the Two Problems:

a. Consider the two problems you have worked on here: the calendar problem and the sequence problem. Is there any similarity between the two problems?
b. Is there a similarity between the strategies you used?

APPENDIX D

'FOUNDATIONS OF MATHEMATICS FOR TEACHERS' PROJECT DESCRIPTION IN ITS ENTIRETY

The project is worth 10% of your final mark. There is a choice of three different projects. Whichever project you choose, it is due on November27th. **Late submissions will not be marked!**

Option 1

"I had been working on the problem for a long time without any progress. Then suddenly I knew the solution, I understood, everything made sense. It seemed like it just CLICKED!"

The above anecdote is a testament of what is referred to as an AHA! experience. Have you ever experienced one? The purpose of this assignment is to have you reflect upon such an AHA! experience and to explore exactly what you learned in that instance and what you think contributed to the moment. You will hand in:

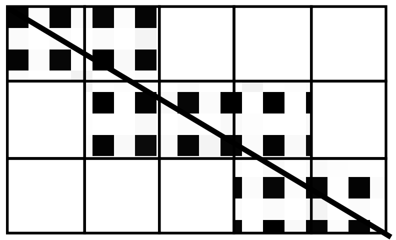
1. A detailed explanation of the specific mathematical topic that you were studying and the difficulty you were having with it (including any incorrect or incomplete understandings that you had of the topic before the AHA!).
2. The story of the AHA! experience as you remember it, paying particular close attention to what you were doing before it happened, when it happened, and how it made you feel when it happened.
3. A detailed explanation of your new understanding of the mathematical topic.
4. A conclusion as to how, upon reflection, the AHA! experience contributes to mathematical learning in general, and for you in particular.
5. Anything else that you feel would contribute to the reader gaining insight into the moment as you experienced it.

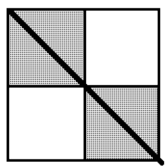
Your final product will be evaluated for completeness and clarity.

Option 2

Consider the following problem:

On a squared paper draw a rectangle and draw in a diagonal.
How many grid squares are crossed by the diagonal?





In the case of a 3x5 rectangle or a 2x2 rectangle above, we can simply count. However, can we make a decision about a 100 x 167 or a 3600 x 288 rectangle? In general, given N x K rectangle, how many grid squares are crossed by its diagonal?

The goal of this assignment is to have you take a closer look at yourself as a learner and a problem solver. You will work in groups of 3 or 4 over the next 4 weeks. Your assignment is to keep a problem solving journal in which you record your attempts to address the problem and your reflection on these attempts.

I suggest that you work on the problem 2-3 times a week for 30-60 minutes, where the last third of your time is spend reflecting on your problem-solving experience.

Your journal should reflect the efforts of the group, so one of the team members needs to take notes on your progress. Your last journal entry should be devoted to the summary of your solution (or partial solution) as well as your analysis of your overall experience with the problem. You may not arrive at a complete solution to the problem. Partial conclusions are welcome.

In marking this project the following will be considered:

1. Evidence of your persistence and willingness to experiment;
2. The presentation of the solution (the thoroughness, organization, clarity of explanation, originality of the solution);
3. Your analysis of your experience

APPENDIX E

"HADAMARD'S SURVEY"[27]

1. At what time, as well as you can remember, and under what circumstances did you begin to be interested in mathematical sciences? Have you inherited your liking for mathematical sciences? Were any of your immediate ancestors or members of your family (brothers, sisters, uncles, cousins, etc.) particularly good at mathematics? Was their influence or example to any extent responsible for your propensity for mathematics?
2. Towards branch of mathematical science did you feel especially attracted?
3. Are you more interested in mathematical science per se or in its application to natural phenomena?
4. Have you a distinct recollection of your manner of working while you were pursuing your studies, when the goal was rather to assimilate the results of others than to indulge in personal research? Have you any interesting information to offer on that point?
5. After having completed the regular course of mathematical studies (which, for instance, corresponds to the program of the Licence mathematique or of two Licences or of the Aggregation) in what direction did you consider it expedient to continue your studies? Did you endeavor, in the first place, to obtain a general and extensive knowledge of several parts of science before writing or publishing anything of consequence? Did you, on the contrary, at first try to penetrate rather deeply into a special subject, studying almost exclusively what was strictly requisite for that purpose, and only afterwards extending your studies little by little? If you have used other methods, can you indicate them briefly? Which one do you prefer?
6. Among the truths which you have discovered, have you attempted to determine the genesis of those you consider the most valuable?
7. What, in your estimate, is the role played by chance or inspiration in mathematical discoveries? Is this role always as great as it appears to be?
8. Have you noticed that, occasionally, discoveries or solutions on a subject entirely foreign to the one you are dealing with occur to you and that these relate to previous unsuccessful research efforts of your?

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[27] Based on Hadamard (1945, pp. 137-141). This version of Hadamard's survey has been edited to update the language and form of the original early 20th century translation.

9. Would you say that your principal discoveries have been the result of deliberate endeavor in a definite direction, or have they arisen, so to speak, in your mind?
10. Have you ever worked in your sleep or have you found in dreams the answers to problems? Or, when you waken in the morning, do solutions which you had vainly sought the night before, or even days before, or quite unexpected discoveries, present themselves ready-made to your mind?
11. When you have arrived at a conclusion about something you are investigating with a view to the publication of your findings, do you immediately write down the part of your work to which your discovery applies; or do you let your conclusions accumulate in the form of notes and begin the redaction of the work only when its contents are important enough?
12. Generally speaking, how much importance do you attach to reading for mathematical research? What advice in this respect would you give to a young mathematician who has had the usual classical education?
13. Before beginning a piece of research work, do you first attempt to assimilate what has already been written on that subject?
14. Or do you prefer to leave your mind free to work unbiased and do you only afterwards verify by reading about the subject so as to ascertain just what is your personal contribution to the conclusions reached?
15. As far as method is concerned, do you make any distinction between invention and redacting?
16. Does it seem to you that your habits of work are appreciably the same as they were before you had completed your studies?
17. When you take up a question, do you try to make as general study as possible of the more or less specific problems which occur to you? Do you usually prefer, first to study special cases or a more inclusive one, and then to generalize progressively?
18. In your principal research studies, have you followed the same line of thought steadily and uninterruptedly to the end, or have you laid it aside at times and subsequently taken it up again?
19. What is, in your opinion, the minimum number of hours during the day, week, or the year, which a mathematician who has other demands on his time should devote to mathematics so as to study profitably certain branches of these same mathematics? Do you believe that one should, if one can, study a little every day, say for one hour at least?
20. Do artistic and literary occupations, especially those of music and poetry, seem to you likely to hamper mathematical invention, or do you think they help it by giving the mind temporary rest?
21. What are your favorite hobbies, pursuits, or chief interests, aside from mathematics, in your leisure time? Do metaphysical, ethical, or religious questions attract or repel you?
22. If you are absorbed by professional duties, how do you fit these into you personal studies?

23. What council, in brief, would you offer to a young man studying mathematics? To a young mathematician who has finished the usual course of study and desires to follow a scientific career?

Questions about daily habits

24. Do you believe that it is beneficial to mathematician to observe a few special rules of hygiene such as diet, regular meals, time for rest, etc.?
25. What do you consider the normal amount of sleep necessary?
26. Would you say that a mathematician's work should be interrupted by other occupations or by physical exercises which are suited to the individual's age and strength?
27. Or, on the contrary, do you think one should devote the whole day to one's work and not allow anything to interfere with it; and, when it is finished, take several days of complete rest? Do you experience definite periods of inspiration and enthusiasm succeeded by periods of depression and incapacity to work? Have you noticed whether these intervals alternate regularly and, if so, how many days, approximately, does the period of activity last and also the period of inertia? Do physical or meteorological conditions (i.e. temperature, light, darkness, the season of the year, etc.) exert an appreciable influence on your ability to work?
28. What physical exercise do you do, or have you done as relaxation form mental work? Which do you prefer?
29. Would you rather work in the morning or in the evening?
30. If you take a vacation, do you spend it studying mathematics (if so, to what extent?) or do you devote the entire time to rest and relaxation?
31. Does one work better standing, seated or lying down? Does one work better standing at the blackboard or on paper? To what extent is one disturbed by outside noises? Can one pursue a problem while walking or in a train? How do stimulants or sedatives (tobacco, coffee, alcohol, etc.) affect the quality and quantity of one’s work?
32. It would be very helpful for the purpose of psychological investigation to know what internal or mental images, what kind of “internal words” mathematicians make use of; whether they are motor, auditory, visual, or mixed, depending on the subject which they are studying.
33. Especially in research thought, do the mental pictures or internal words present themselves in the full consciousness or in the fringe-consciousness? The same question is asked concerning the arguments which these mental pictures or words may symbolize?

APPENDIX F

SURVEY SENT TO MATHEMATICIANS IN ITS ENTIRETY

Dear Professor ,

I am a doctoral student working on my dissertation in mathematics education and I am asking for your help. I am attempting to reproduce part of Hadamard's classic survey (see below). The particular questions to which I'm most interested in getting answers are:

1. Would you say that your principle discoveries have been the result of deliberate endeavour in a definite direction, or have they arisen, so to speak, spontaneously? Have you a specific anecdote of a moment of insight/inspiration/illumination that would demonstrate this? [Hadamard # 9]

2. How much of mathematical creation do you attribute to chance, insight, inspiration, or illumination? Have you come to rely on this in any way? [Hadamard# 7]

3. Could you comment on the differences in the manner in which you work when you are trying to assimilate the results of others (learning mathematics) as compared to when you are indulging in personal research (creating mathematics)? [Hadamard # 4, 15]

4. Have your methods of learning and creating mathematics changed since you were a student? How so? [Hadamard # 16]

5. Among your greatest works have you ever attempted to discern the origin of the ideas that lead you to your discoveries? Could you comment on the creative process that lead you to your discoveries? [Hadamard # 6]

I would greatly value your comments on these five questions in particular, but responses to any of the five or any of the questions from the original survey (attached) would be greatly appreciated.

Hadamard's Survey: Background

In 1943 Jacques Hadamard gave a series of lectures on mathematical invention at the Ecole Libre des Hautes Etudes in New York City. These talks were subsequently published as The Psychology of Mathematical Invention in the Mathematical Field (Princeton University Press). Hadamard's seminal work outlines the beliefs of contemporary mathematicians as to the mechanism by which they come to create new mathematics. There exists no such work representing the beliefs of modern mathematicians.

I have made this the focus of my doctoral work in mathematics education. Working under the supervision of Peter Borwein (Professor of Mathematics, SFU), Rina Zazkis (Professor Education, SFU) and Tom O'Shea (Professor of Education, SFU) I have chosen to revisit Hadamard's classic survey in the hope of gaining some insight into the creative process associated with doing mathematics.

Hadamard based his survey on an earlier one that was published in the periodical L'Enseignement Mathematique (Vol. I, 1902 and Vol. VI, 1904). He was critical of this earlier survey in that it failed to solicit the views of "mathematicians whose creative processes are worthy of interest" (Hadamard, p.10). Hadamard set excellence in the field of mathematics as a criterion for participation in his study. In keeping with Hadamard's standards I have chosen to survey the most prominent mathematicians (Fields medallists, Nevanlinna Prize winners, and members of: The Royal Society, American Society of Arts & Sciences, and the Academie des Sciences). As a member of at least one of the aforementioned groups I seek your help.

My intention is to use this data as the basis for my research on the creative nature of doing mathematics. This will serve to bolster my argument that mathematics instruction and curriculum needs to more seriously consider this too often overlooked aspect of mathematics. As such, it may be necessary to quote some or all of your responses within my dissertation. In addition, as who you are lends credence to what you say I would like to use your name in association with some of your comments. By responding to the survey I will assume that you are willing to participate in my research and are willing to allow me to anonymously quote some or all of your comments. However, I ask that if you would permit me to use your name in association with your comments that you state so explicitly within your response.

Thank you for your time and support. If you have any concerns regarding this research you may contact Director, Office of Research Ethics by email at hweinber@sfu.ca or phone at 604-268-6593. If you have any other questions or comments please feel free to contact me at pgl@sfu.ca.